

# GENERALIZED SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS A NONSTANDARD JETS APPROACH

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ABSTRACT. Using the rudiments of pde jets theory in a nonstandard setting, we first deepen and extend previous nonstandard existence results for generalized solutions of linear differential equations and second extend the previous results for linear differential equations to a much broader class of nonlinear differential equations.

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ABSTRACT. We extend a recently proved result of Todorov asserting the existence of generalized solutions of very general linear partial differential operators. To linear operators whose symbols vanish only to finite order, we prove existence of solutions to infinite order on  ${}^\sigma\mathbb{R}^m$ . We prove existence of generalized solutions for nonlinear operators satisfying  ${}^\sigma PCP$ , a condition that implies Todorov's condition in the linear case. In the conclusion, we prove that generalized solutions on  ${}^\sigma\mathbb{R}^m$  are remarkably abundant .

## 1. INTRODUCTION

In this paper, we extend the results of Todorov [27] (on the existence of generalized solutions for a general set of differential operators) in two directions. If  $P$

is a linear partial differential operator of order  $r$ , written as  $P \in LPDO(r)$ , with  $C^\infty$  coefficients, and  $\lambda_P$  is its (total) symbol, then Todorov proves the existence of generalized solutions  $f$  for the equation  ${}^*P(f)({}^*x) = {}^*g({}^*x)$  for all  $x \in \mathbb{R}^m$  outside  $\mathcal{Z}_{\lambda_P}$  and for quite general  $g$ , where  $\mathcal{Z}_h$  denotes the  $\{x \in \mathbb{R}^n : h(x) = 0\}$ . From a slightly different perspective, Todorov's result says that for a general set of standard  $g$ , there exists internal  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , such that  $({}^*P(f) - {}^*g)|_{\sigma\mathbb{R}^m} = 0$ . ( $\sigma\mathbb{R}^m$  denotes the standard vectors in the internal vector space  ${}^*\mathbb{R}^m$ ). That is,  ${}^*P(f) - g$  vanishes pointwise, ie., has  $0^{th}$  order contact with  ${}^*\mathbb{R}^n$  at each point of  $\sigma\mathbb{R}^m$ . The standard geometry and jet definitions with respect to PDEs (partial differential equations) are recalled in the next section. The first extension of Todorov in this paper is to give a straightforward construction that there exists internal smooth maps  $f$  such that  ${}^*P(f) - {}^*g$  has infinite order contact with  ${}^*\mathbb{R}^n$  at all points of  $\sigma\mathbb{R}^m$ . (See Theorem 5.1.) Another way of stating this is that Todorov solves the equations at each standard  $x$ ; here we solve the equations along with the infinite family of integrability differential equations associated to the given differential equation at each such  $x$ . But the two corollaries carry the critical import of this theorem. Corollary 5.3 is the infinite order direct descendant of Todorov's result. It depends on the standard jet space work done in Section 4. The needed standard statement following from this work lies in Corollary 4.1. The second corollary becomes possible only within the perspective of this paper. We can consider those partial differential operators whose (total) symbols vanish to some finite order, ie., have any finite order contact with  ${}^*\mathbb{R}^m$  along  $\sigma\mathbb{R}^m$ ; see Section 3. Note here that Todorov only consider the  $0^{th}$  order vanishing case. Our Corollary 5.4 says that as long as the vanishing order of  ${}^*g$  at standard points is controlled by that of the symbol of  ${}^*P$ , we can find internal smooth  $f$  solving  ${}^*P(f) - {}^*g = 0$  to infinite order on  $\sigma\mathbb{R}^m$ . Such a theorem is unstable within the venue of Todorov's setting. The work in standard geometry allowing the proof of this result occurs in section 3; see Corollary 3.1. Parenthetically, it's conceivable that the PDE jet results of Sections 3 and 4 exist in the literature, but the author could not find them.

The overwhelming bulk of the work in this paper concerns the linear theory. But, the nonlinear PDE jet framework and NSA fit quite well together, and so the second direction of extension of the result of Todorov is into nonlinear partial differential equations, NLPDEs. There is a well developed theory of nonlinear partial differential equations within the jet bundle framework, exemplified in the texts of Pommaret, [22] Olver, [21] and Vinogradov, [12]. Nonstandard analysis is as comfortable in this framework as in the linear. So, in Section 6, we introduce simple conditions,  $PCP$ , and  $\sigma PCP$ , on the symbols of general NLPDEs of finite order, and give an easy proof of existence of generalized solutions in the sense of Todorov for those NPDO's satisfying these criteria. We show that Todorov's nonvanishing condition on the symbol implies that his  $LPDO$ 's satisfy  $\sigma PCP$ . But, our theorem asserts the existence of generalized solutions in the far broader nonlinear arena.

The standard import of these results is yet to be worked out. See the conclusion for a curious result on this. We prove a result that might appear startling: almost all internal smooth functions are solutions on  $\sigma\mathbb{R}^m$  of any standard differential operator that has the zero function as a solution. It seems that the work of Baty, et al., [3], might be a useful framing for this. That is, their analysis needs a lot of elbow room on the infinitesimal level to allow adjusting eg., the infinitesimal widths of Heaviside jumps, etc. It seems that the results here might be interpreted

as saying that the formal (nonstandard) jet theory of symbols allows such roominess for such empirically motivated adjustments.

The author relies on a jet bundle framework when some might consider it too big a machine for the job. Yet, from the point of view of nonstandard analysis, the jet bundle framework is natural and eg., allows an easy generalization of Todorov's result to the nonlinear case. The total and principal symbols of a differential operator have a natural geometric setting which when extended to the nonstandard world allows a geometric consideration of generalized solutions and, in fact generalized differential operators vis  $^*$ smooth symbols.

Todorov defines his differential equation and constructs his solutions within spaces of generalized functions defined on  $^*\Omega$ , where  $\Omega$  an arbitrary open subset of  $\mathbb{R}^m$ , and gets his localizable differential algebra of generalized functions by 'quotienting' out by the parts of the  $^*C^\infty$  functions defined on nonnearstandard points of  $^*\Omega$ . (Note also his NSA jazzed up version of the constructions of Colombeau, Oberguggenberger, and company in eg., [25], of which we will say more later). On the other hand, my paper focuses almost exclusively on extending Todorov's existence result to more general classes of differential operators and very little on a broader analysis of his differential algebra of generalized functions. (In a follow up to this paper, we will refine the results appearing here within the aforementioned nonstandard version of Colombeau's algebra of generalized functions constructed by Todorov and Oberguggenberger, [20]). Accordingly, our constructions occur on all of  $^*\mathbb{R}_{nes}^m$ . If we restrict to differential operators whose finite vanishing order sets don't have infinitely many components so that they have no nontrivial  $^*$ limiting behavior at nonnearstandard points, the results here should hold without change for Todorov's localizable differential algebras.

The geometric theory of differential equations and their symmetries, as exemplified in eg., Olver, [21] and Pommaret, [22]. is a natural framework within which to integrate the generalizing notions of NSA. This is the first of a series of papers in which the author intends to attempt a theory of generalized solutions (existence and regularity) and symmetries of differential equations within the context of the extensive jet theory. Note that although this approach seems to be new, there are a growing number of research programs moving beyond classical approaches; see eg., Colombeau, [7], Oberguggenberger, [18] and Rossinger, [23]. For a good overview of the new theories of generalized functions, see eg., Hoskins and Pinto, [11], and for specific surveys of the obstacles to the construction of a nonlinear generalization of distributions and a comparison of the characteristics of the these new theories, see Oberguggenberger, [19] and more recently Colombeau, [6]. Note, in particular the flurry of work extending the arena of Colombeau algebras into mathematical physics involving differential geometry and topology that Kunzinger, [13], summarizes.

Further note that Oberguggenberger and Todorov, see eg., [20], have shown how much of the theoretical foundations of Schwarz type Colombeau algebras can be simplified and strengthened within the venue of nonstandard constructions and recent work of Todorov and coworkers, see eg., [25], have extended the results with this model. See also Todorov's lecture notes, [26]. None of these approaches consider the symbol of the differential operator, and the nonstandard extension of its geometric milieu as the primary object of study. This is the perspective of the current work. Finally, it seems that nonstandard methods are much more encompassing than the impressive work of the Colombeau school of generalized

functions; eg., consider the work of the mathematical physicists working around Baty, eg., see [3] and [2]. Note, in particular the perspective of Baty, et al on p37 of [3] (with respect to the benefits of nonstandard methods) where they note that the generalized functions of the Columbeau school

...are not smooth functions and do not support all of the operations of ordinary algebra and calculus, the multiplication of singular generalized functions is accomplished via a weak equality called association. In contrast to such calculations, the objects manipulated in equations (4.4) to (4.13) (and indeed in the following section of this report) are smooth nonstandard functions.

In reading their papers, it's clear that their need for "ordinary algebra", etc., is critical to their analysis. The author also believes that, here also, having at hand the full capacity of mathematics via transfer straightforwardly allows many of the constructions of this paper.

## 2. SOME JET PDE BASICS AND NONSTANDARD VARIATIONS

### 2.1. Nonstandard analysis.

2.1.1. *Resources.* Good introductions to nonstandard analysis abound. One might start with the pedestrian tour of its basics in the introduction of the authors dissertation, [16]; then get deeper with the introduction of Lindström, [14] and follow this with the constructive introduction of Henson, [10]. There is also Nelson's (axiomatic) internal set theory, a theory with similar goals and achievements to Robinson's (currently superstructure/ultrapower) nonstandard analysis; a good text being that of Lutz and Goze, [15]. One could also check out strategic outgrowths from these nonstandard schools, eg., the work of Di Nasso and Forti, [8]. One might also check out their (and Benci's) constructive survey, [5], of a variety of means to a nonstandard mathematics (which also includes a good introduction). There are yet other approaches to a nonstandard mathematics, notably the Russian school; for a good example, see [9].

2.1.2. *Impressionistic introduction with terminology.* Let's give an impressionistic introduction to nonstandard mathematics via the (extended) ultrapower constructions. There are many motivations for the need of a nonstandard mathematics. To have our real numbers,  $\mathbb{R}$ , embedded in a much more robust object,  ${}^*\mathbb{R}$ , with the properties of the real numbers, but also containing infinite and infinitesimal quantities is a boon to a direct formalization of intuitive strategies. Model theoretically, these have been around for more than 60 years (some would argue much longer) via eg., ultrapowers or the compactness theorem.

The ultrapower is the construction generally least involved with theoretical matters of the foundations of math (but see [8]). Thinking of eg., infinite numbers as limiting properties of sequences of real numbers, one might attempt to construct nonstandard real numbers as equivalence classes of such sequences, ie.,  ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \sim$  where  $/ \sim$  denotes the forming of such equivalence classes. Clearly, one can extend the operations and relations on  $\mathbb{R}$  to  $\mathbb{R}^{\mathbb{N}}$  coordinate wise, getting a partially ordered ring; but almost all of the nice properties of  $\mathbb{R}$  are lost. But, if  $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ , the power set of  $\mathbb{N}$ , it turns out that there are objects  $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ , the ultrafilters, such that defining our equivalence relation in terms of elements of  $\mathcal{U}$  preserves all "well stated" properties of  $\mathbb{R}$ . More specifically, given  $(r_i), (s_i) \in \mathbb{R}^{\mathbb{N}}$ , define  $(r_i) \sim (s_i)$

if  $\{i : r_i = s_i\} \in \mathcal{U}$  and  $(r_i) < (s_i)$  if, again  $\{i : r_i < s_i\} \in \mathcal{U}$ , we find that the extended ring operations and partial order are well defined on the quotient  $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$  and, in fact, it's not hard to prove that we get a totally ordered field containing  $\mathbb{R}$  (the set of equivalence classes of constant sequences) as a subfield. For  $r \in \mathbb{R}$ , let  ${}^*r = (r)/\sim$ , the equivalence class containing the corresponding constant sequence and  ${}^\sigma A = \{{}^*r : r \in A\} \subset {}^*\mathbb{R}$  denote the image of  $A \subset \mathbb{R}$  in our nonstandard model of  $\mathbb{R}$ ; eg.,  ${}^\sigma \mathbb{R}$  is the image of  $\mathbb{R}$ . In general, we will let  $\langle r_i \rangle$  denote the equivalence class of a sequence  $(r_i) \in \mathbb{R}^{\mathbb{N}}$ . Given this, existence of infinite elements is clear: if  $(r_i) \in \mathbb{R}^{\mathbb{N}}$  with  $r_i \rightarrow \infty$  as  $i \rightarrow \infty$ , then for each  $s \in \mathbb{R}$ , the set  $\{i : r_i > s\}$  is in  $\mathcal{F}(\subset \mathcal{U})$  and so, by our definition,  $\langle r_i \rangle > {}^*s$ . Note that to verify the field properties and total ordering of  ${}^*\mathbb{R}$ , we need the full strength of ultrafilters, eg., the maximality property: if  $A \subset \mathbb{N}$ , then precisely one of  $A$  or  $\mathbb{N} \setminus A$  is in  $\mathcal{U}$ . In particular, if  $\omega = \langle m_i \rangle \in {}^*\mathbb{N}$  where  $m_i \uparrow \infty$  as  $i \rightarrow \infty$ , eg.,  $\omega$  is infinite

Since we can form the  $\mathcal{U}$  equivalence class of arbitrary sequences of real numbers and get a much larger field with all of the 'same' well formed properties as  $\mathbb{R}$ , why can't we do this for  $\mathbb{N}$ ,  $\mathbb{Q}$ ,  $\mathbb{Q}(\sqrt{5})$ , etc. and get 'enlarged' versions of these? We can, but try to do this with the algebra  $F(\mathbb{R})$  of real valued smooth maps on  $\mathbb{R}$ ; ie., consider  $F(\mathbb{R})^{\mathbb{N}}/\sim$  as before. Clearly, this is a ring, but do these 'functions' (on  ${}^*\mathbb{R}$ ) have, in some good sense, all of the properties of functions in  $F(\mathbb{R})$ ? Ignoring subtleties, the simple answer is yes, simply because these elements are internal and therefore fall under the aegis of the all encompassing *principle of transfer*; but let's see, to some extent, how this works in this case. Let's consider, for example, the nonstandard support ( ${}^*$ support) of an equivalence class  $\langle f_i \rangle \in {}^*F(\mathbb{R})$ . (Recall that if  $f \in F(\mathbb{R})$ , then the support of  $f$ ,  $\text{supp}(f)$ , is the closure of the set of  $t \in \mathbb{R}$  where  $f(t) \neq 0$ .) But, then as we seem to be following a recipe of extending everything component wise and then taking the quotient, if  $A_i = \text{supp}(f_i)$ , then  ${}^*\text{supp}(\langle f_i \rangle)$  must be the equivalence class  $\langle A_i \rangle$ .

Yet, how is this a subset of  ${}^*\mathbb{R}$ ? This is a special case of the next problem of nonstandard analysis: extending 'is an element of' to our ultrapower constructions. Miraculously, the properties of ultrafilters (eg., our  $\mathcal{U}$ ) allow one to (simplymindedly!) define  $\langle r_i \rangle \in \langle A_i \rangle$  if  $\{i : r_i \in A_i\} \in \mathcal{U}$ . (This really should be written  $\langle r_i \rangle {}^*\in \langle A_i \rangle$ , but starring all extended operations, relations, etc. can rapidly get confusing.) Note that these subsets of  $\mathcal{P}({}^*\mathbb{R})$  of the form  $\langle A_i \rangle$  are called *internal sets* and are *precisely those subsets that extend the properties of  $\mathcal{P}(\mathbb{R})$  (and therefore shall be denoted  ${}^*\mathcal{P}(\mathbb{R})$ ) via the principle of transfer*. For example, the typical bounded subset  $\mathcal{C}$  of  ${}^*\mathbb{R}$  does not have a nonstandard supremum, ie.,  ${}^*\sup \mathcal{C}$  does not exist; in particular, the transfer principle applied to the theorem that bounded subsets of  $\mathbb{R}$  have suprema does not transfer to all  ${}^*$ bounded elements of  $\mathcal{P}({}^*\mathbb{R})$ . (For example, the set of *infinitesimals*, denoted  $\mu(0)$  here, certainly does not have a supremum.) Nonetheless, the transfer principle certainly does apply to the internal  $\langle A_i \rangle$ , and the proper definition is (surprise!)  ${}^*\sup \langle A_i \rangle = \langle \sup A_i \rangle$ . Note here that other notable examples of external subsets of  ${}^*\mathbb{R}$  are  ${}^\sigma \mathbb{R}$  (and in fact  ${}^\sigma A$  for any infinite subset  $A \subset \mathbb{R}$ ),

$$(1) \quad {}^*\mathbb{R}_{nes} = \{t \in {}^*\mathbb{R} : |t - {}^*s| \in \mu(0) \text{ for some } s \in \mathbb{R}\},$$

the *nearstandard real numbers*,  ${}^*\mathbb{R}_\infty$ , the infinite real numbers and the infinite natural numbers,  ${}^*\mathbb{N}_\infty$  which are said to be *\*finite*. If  $A \subset \mathbb{R}$  is finite, let  $|A| \in \mathbb{N}$  denote its cardinality, let  $\omega = \langle m_i \rangle \in {}^*\mathbb{N}$  be an infinite  ${}^*$ finite integer and  $A_i \subset \mathbb{R}$  be such that  $\{i : |A_i| = m_i\} \in \mathcal{U}$ . Then we say that  $\langle A_i \rangle \subset {}^*\mathbb{R}$  is a *\*finite subset of*

*\*cardinality*  $\omega$ . Although (for  $\omega$  infinite) these sets are infinite (in fact uncountable!), transfer implies that *\*finite* subsets of  $\mathbb{R}$  have the ‘same’ properly stated properties that finite subsets have. Nonetheless, for a much stronger *sufficiently saturated* ultrapowers, there exists *\*finite* subsets of  ${}^*\mathbb{R}$  containing  ${}^\sigma\mathbb{R}$ . These will play a role in this paper.

We still haven’t considered how the elements of  ${}^*F(\mathbb{R}) = F(\mathbb{R})^{\mathbb{N}} / \sim$  can be considered as functions on  ${}^*\mathbb{R}$ , but by now the reader can see that we must define  $\langle f_i \rangle(\langle x_i \rangle) = \langle f_i(x_i) \rangle$  and hope that the properties of  $\mathcal{U}$  ensure that this is well defined (ie., independent of choice of representatives) and is a function. This can indeed be verified and these functions are the *internal functions* in  $F({}^*\mathbb{R})$ ; the function  $f : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$  defined by  $f(x) = x$  if  $x \sim 0$ , ie., if  $x$  is infinitesimal, and  $f(x) = 0$  if  $x \not\sim 0$  is an *external function*, eg., does not satisfy the internality criteria allowing the use of transfer. For example, it is *\*bounded* (bounded in  ${}^*\mathbb{R}$ ), but  ${}^*\sup f$  does not exist. Yet again, it’s straightforward that for *\*bounded*  $\langle f_i \rangle$ ,  ${}^*\sup \langle f_i \rangle$  is well defined precisely by our recipe:  $\langle \sup f_i \rangle \in {}^*\mathbb{R}$  (this *\*supremum* may be an infinite element of  ${}^*\mathbb{R}$ ). Internal subsets of  ${}^*\mathbb{R}$  of the form  $\langle A \rangle$  (ie., the equivalence class containing the constant sequence  $(A_i)$  for some  $A \subset \mathbb{R}$ ) are called the *standard sets*. Following our recipe for denoting the equivalence class of a constant sequence by starring,  $\langle A \rangle$  is usually denoted  ${}^*A$ . For perspective, note that the copy of  $[0, 1]$  lying in  ${}^*[0, 1]$ , ie.,  ${}^\sigma[0, 1] \subset {}^*[0, 1]$  is very sparse. For example, given an infinitesimal,  $0 < t = \langle t_i \rangle \in {}^*\mathbb{R}$  (eg., suppose  $t_i \downarrow 0$  as  $i \rightarrow \infty$ ) and  $r \in (0, 1)$ , then  ${}^*r + [-t, t] \subset {}^*[0, 1]$ , but intersects  ${}^\sigma[0, 1]$  only at  ${}^*r$ . Let’s consider the *standard function*  ${}^*\sin(x)$ . First of all,  ${}^*\sin$  is defined essentially as we defined standard sets,  ${}^*A$ , ie., the  $f_i$  above are all the function  $\sin$ . So if we define the *\*domain* of  $\langle f_i \rangle$  as we have all else:  ${}^*\text{dom}(\langle f_i \rangle) \doteq \langle \text{dom}(f_i) \rangle$ , we see that  ${}^*\text{dom}({}^*\sin)$  is all of  ${}^*\mathbb{R}$ . (Or, as the domain of  $\sin$  is  $\mathbb{R}$ , transfer says that the *\*domain* of  ${}^*\sin$  is  ${}^*\mathbb{R}$ .) A consequence of our constructive approach is the fact that  ${}^*\text{dom}({}^*\sin) = \langle \text{dom}(f_i) \rangle$  is internal. It’s not hard to check that  ${}^*\sin$  is really an extension of  $\sin : \mathbb{R} \rightarrow [-1, 1]$ : first, restricting the graph of  ${}^*\sin$  to  ${}^\sigma\mathbb{R}$  just the image of the graph of  $\sin$  in  ${}^*\mathbb{R}^2$ ; second, all of the symmetry and character properties hold and third, it has all of the (transferred) analytic properties that  $\sin$  has.

Before we conclude this tour, let’s look at the standard part map. We defined the external (subring)  ${}^*\mathbb{R}_{nes} \subset {}^*\mathbb{R}$  above. It should not be surprising that this is precisely those  $\langle r_i \rangle \in {}^*\mathbb{R}$  satisfying  $|\langle r_i \rangle| < {}^*t$  for some  $t \in \mathbb{R}$  (here  $|| : {}^*\mathbb{R} \rightarrow {}^*[0, \infty)$  is defined as all else). But by it’s definition, any  $\langle r_i \rangle \in {}^*\mathbb{R}_{nes}$  satisfies  $\langle r_i \rangle \sim {}^*r$  for some (clearly unique)  $r \in \mathbb{R}$ , eg., there is a well defined map (homomorphism onto!)  $\text{st} : {}^*\mathbb{R}_{nes} \rightarrow \mathbb{R}$ , the *standard part map*. Sometimes we will write  ${}^o\langle r_i \rangle$  for  $\text{st}\langle r_i \rangle$ . Note then that if  $\langle f_i \rangle : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$  has image in  ${}^*\mathbb{R}_{nes}$ , then we can define  $\text{st}\langle f_i \rangle : \mathbb{R} \rightarrow \mathbb{R}$  to be the map  $r \in \mathbb{R} \mapsto \text{st}(\langle f_i \rangle({}^*r))$ . Given this, if  $\omega = \langle m_i \rangle \in {}^*2\mathbb{N}$  with  $m_i \uparrow \infty$  as  $i \rightarrow \infty$  (eg.,  $\omega$  is infinite), consider  $f_i$  given by  $x \mapsto \sin(m_i x)$ , so that writing  $\xi = \langle x_i \rangle \in {}^*\mathbb{R}$ , we have  ${}^*\sin(\omega\xi) = \langle \sin(m_i x_i) \rangle$ . By transfer,  $\xi \mapsto {}^*\sin(\omega\xi)$  has all of the symmetry and analytic properties of  $x \mapsto \sin(2mx)$  for some  $m \in \mathbb{N}$ , eg., solves the nonstandard *\*differential equation*  $f'' - \omega^2 f = 0$ ; yet it’s standard part is not even Lebesgue measurable!

**2.1.3. Formal tools.** The four pillars of nonstandard analysis are the internal definition principle, transfer, saturation and (several versions of) “overflow”. In order to discern the internal sets among all external sets, one can use the internal definition

principle. It is basically an algorithmic way of determining if some object is of the form  $\langle S_i \rangle$  and depends on the fact that all internal sets are elements of some standard set  ${}^*T$ . It asserts that if  $\mathcal{B} = \langle B_i \rangle$  is internal and  $P(H)$  is a statement about an variable quantity  $H$  in an internal set  $\mathcal{X}$  (of  ${}^*$ functions,  ${}^*$ measures, etc.), then  $\{H \in \mathcal{X} : P(H) \text{ is true}\}$  is internal. As described above transfer allows us to, eg., translate to the nonstandard world careful statements about regular mathematics. Here we will need it to, eg., transfer to the nonstandard world the existence of maps of a certain type that have specified values on finite sets. Next, saturation has a variety of guises, one of which will be important here. Besides the need for the monads associated with neighborhood filters for a given topology, the specific form of saturation (see Stroyan and Luxemburg, [24], chapter 8) that will be needed here, in section 4, ensures that  ${}^*$ finite set are sufficiently large. Specifically, let  $X \in \mathcal{U}$  be an infinite set of cardinality not bigger than  $\mathcal{P}(C^\infty(\mathbb{R}^m, \mathbb{R}))$ . Then, there exists a  ${}^*$ finite  $\mathcal{A} \in \mathcal{U}$  with  ${}^\sigma X \subset \mathcal{A} \subset {}^*X$ . This can be situated so that  $\mathcal{A}$  carries the same, well formed finitely stated, characteristics as  $X$  (transfer). We will use this in the situation where  $X$  is a particular collection of smooth maps or smooth section of a bundle. We will also use an overflow type result that depends on our nonstandard model being polysaturated. See [24] chapter 7; below we paraphrase their theorem 7.6.2 for our use.

**Theorem 2.1.** *Suppose that  $A \subset {}^*\mathbb{R}^m$ , not necessarily internal with cardinality less than that of  ${}^*\mathbb{R}^n$ . Suppose that  $F : A \rightarrow {}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  is any map. Then there is an internal subset  $\bar{A} \subset {}^*\mathbb{R}^m$  and an internal map  $\bar{F} : \bar{A} \rightarrow {}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  such that  $A \subset \bar{A}$  and  $\bar{F}|_A = F$ .*

**2.2. Jet bundle constructions.** In this section we cover enough of the basics of jets and the jet bundle formulation of (linear) differential operators, sufficient to formulate and prove our results.

**2.2.1. Jet bundle setup.** We will briefly summarize that part of jet theory that we need. Although the following formulation is straightforwardly generalized to smooth manifolds, for brevity's sake we will restrict to the Euclidean case. Let  $\mathbf{P}_k(\mathbf{m}, \mathbf{n})$  denote the polynomial maps of order  $k$  from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . If  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$ ,  $x \in \mathbb{R}^m$  and  $k \in \mathbb{N}$ , let  $\mathbf{T}_x^k \mathbf{f} \in P_k(m, n)$  denote the  $k$ th order Taylor polynomial of  $f$  at  $x$ . By  ${}^*$  transfer, if  ${}^*P_k(m, n)$  denotes the  ${}^*\mathbb{R}$  vector space of internal polynomials from  ${}^*\mathbb{R}^m$  to  ${}^*\mathbb{R}^n$ ,  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  and  $x \in {}^*\mathbb{R}^m$ , we have  ${}^*T_x^k f$ , the internal  $k^{th}$  order Taylor polynomial of  $f$  at  $x$ . Note that transfer implies that this has all of the properties of the Taylor polynomial, suitably interpreted. Note that although  $f = {}^*g$  so that  $\xi \mapsto {}^*T_\xi^k f$  is simply the transfer of the standard map  $x \mapsto T_x^k f$ , for  $\xi \in {}^*\mathbb{R}_\infty^m$  or  $k \in {}^*\mathbb{N}_\infty$ ,  ${}^*T_\xi^k f$  can be very pathological.

We define an equivalence relation on  $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  as follows. We say that  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  **vanishes to  $k$ th order at  $x$**  if  $T_x^k f = 0$  and for  $f, g \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ , we say that  $f$  equals  $g$  to  $k$ th order, written  $\mathbf{f} \stackrel{x_k}{\sim} \mathbf{g}$  if  $T_x^k(f - g) = 0$ . This defines an equivalence relation on  $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ . Let  $\mathbf{j}_x^k \mathbf{f}$  denote the equivalence class containing  $f$ . We denote the set of equivalence classes by  $\mathcal{J}_{\mathbf{m}, \mathbf{n}, \mathbf{x}}^k$ . There are a variety of definitions of  $\mathcal{J}_{\mathbf{m}, \mathbf{n}, \mathbf{x}}^k$  and its not hard to show that one can identify  $\mathcal{J}_{\mathbf{m}, \mathbf{n}, \mathbf{x}}^k$  with the set of Taylor polynomials of order  $k$  at  $x$  of smooth maps  $(\mathbb{R}^m, x) \rightarrow \mathbb{R}^n$  and we can identify  $\mathbf{j}_x^k \mathbf{f}$  with  $T_x^k f$ . (An equivalence class consists of all maps with a given  $k^{th}$  order Taylor polynomial at  $x$ .) Let  $\mathcal{J}_{\mathbf{m}, \mathbf{n}}^k = \cup_{x \in \mathbb{R}^m} \mathcal{J}_{\mathbf{m}, \mathbf{n}, \mathbf{x}}^k$ .  $\mathcal{J}_{\mathbf{m}, \mathbf{n}}^k$  is a

smooth fiber bundle, in fact, as our maps have range  $\mathbb{R}^m$ , a vector bundle, over  $\mathbb{R}^m$  with fiber over  $x \in \mathbb{R}^n$  given by  $\mathcal{J}_{m,n,x}^k$ . Let  $\pi_0^k : \mathcal{J}_{m,n}^k \rightarrow \mathbb{R}^m$  denote the bundle projection. Note also that if  $l, k \in \mathbb{N}$  with  $l > k$ , then  $\mathcal{J}_{m,n}^l$  is a bundle over  $\mathcal{J}_{m,n}^k$ ; let  $\pi_k^l$  denote the bundle projection. We let  $C^\infty(\mathcal{J}_{m,n}^k)$  denote the  $\mathbb{R}$  vector space of  $C^\infty$  sections of  $\mathcal{J}_{m,n}^k$ . If  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ , there is a canonical section of  $\pi_0^k$  given by  $j^k f : x \mapsto j_x^k f$ . There is a canonical map, the operation of taking the  $k$  jet:

$$(2) \quad j^k : C^\infty(\mathbb{R}^m, \mathbb{R}^n) \rightarrow C^\infty(\mathcal{J}_{m,n}^k) \quad f \mapsto j^k f$$

For later purposes we also need to define the infinite jet,  $j_x^\infty f$ , for  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ . Doing a simplified rendering of projective limits, we will define the vector space of infinite jets at  $x \in \mathbb{R}^m$ ,  $\mathcal{J}_{m,n,x}^\infty$ , to be the set of sequences  $(f^0, f^1, f^2, \dots)$  such that  $f^k \in \mathcal{J}_{m,n,x}^k$  for all  $k$  and for all nonnegative integers  $j < k$ ,  $\pi_j^k(f^k) = f^j$ . Then for a given  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ , the infinite jet of  $f$  at  $x$ ,  $j_x^\infty f$ , is clearly the well defined element of  $\mathcal{J}_{m,n,x}^\infty$  given by  $(j_x^0 f, j_x^1 f, j_x^2 f, \dots)$ . It is easy to see that  $\mathcal{J}_{m,n,x}^\infty$  is an infinite dimensional vector space over  $\mathbb{R}^m$ , operations given componentwise, and that, for each  $x \in \mathbb{R}^m$ , the map  $j_x^\infty : f \mapsto j_x^\infty f : C^\infty(\mathbb{R}^m, \mathbb{R}^n) \rightarrow \mathcal{J}_{m,n,x}^\infty$  is a vector space surjection with kernel the subspace of  $g \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  such that  $j_x^k g = 0$  for all integers  $k$ , ie.,  $g$  vanishes to infinite order at  $x$ . We will also need the forgetful fiber projection  $\pi_{k,x} : \mathcal{J}_{m,n,x}^\infty \rightarrow \mathcal{J}_{m,n,x}^k$ . As the base range space is linear,  $\pi_{k,x}$  is a surjective linear morphism and clearly has kernel the (ideal) of formal power series that vanish to  $k^{th}$  at  $x$ , see above.

From the canonical (global) coordinate framing  $x_i$ ,  $1 \leq i \leq m$  on  $\mathbb{R}^m$ , and  $y^j$ ,  $1 \leq j \leq n$  on  $\mathbb{R}^n$ , we get induced coordinates,  $x_i, y_\alpha^j$  for  $|\alpha| \leq k$  and  $1 \leq j \leq n$ , on  $\mathcal{J}_{m,n}^k$  defined as follows. The  $x_i$  are just the pullback of the coordinates on the base  $\mathbb{R}^m$ . If  $\phi \in \mathcal{J}_{m,n}^k$ , then  $\phi \in \mathcal{J}_{m,n,x_0}^k$  for some  $x_0 \in \mathbb{R}^m$  and so  $\phi$  can be written as  $j_{x_0}^k f$  for some  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ . Then

$$(3) \quad x_i(\phi) = x_{0,i}, \quad y_\alpha^j(\phi) \doteq \partial^\alpha(f^j)(x_0) = \phi_\alpha^j.$$

where  $\partial^\alpha$  denotes  $\frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \dots (\partial x_m)^{\alpha_m}}$  where  $\alpha = (\alpha_1, \dots, \alpha_m)$ . If  $\lambda \in C^\infty(\mathcal{J}_{m,n}^k, \mathbb{R}^n)$ , then with respect to the induced coordinates, we write this as  $\lambda(x_i, y_\alpha^j)$ . therefore, for later use, we can Taylor expansion  $\lambda$  **with respect to the  $x_i$  coordinates**, around a given  $p_0$ , as follows. Let  $p_0 = (x_{0,i}, y_{0,\alpha}^k)$  be a coordinate representation as above. Let  $\lambda^l$  denote the  $l^{th}$  coordinate of  $\lambda$  with respect to the canonical coordinates on  $\mathbb{R}^n$ . Then the Taylor expansion to order  $s$  with in the base coordinates is

$$(4) \quad \lambda^l(x_i, y_\alpha^k) = \sum_{|\alpha| \leq s} K_\alpha (x - x_0)^\alpha \partial^\alpha(\lambda^l)(p_0) + \tilde{\lambda}^l(x_i, y_\alpha^k)$$

where for  $|\alpha| = s+1$ ,  $\tilde{\lambda}^l \in C^\infty(\mathcal{J}_{m,n}^k, \mathbb{R}^n)$  vanishes to order  $s+1$  in the base coordinates at  $p_0$ ,  $K_\alpha$  are the usual factorial constants,  $(x - x_0)^\alpha = (x_1 - x_{0,1})^{\alpha_1} \dots (x_m - x_{0,m})^{\alpha_m}$ , and  $\partial^\alpha$  is the  $\alpha^{th}$  partial derivative with respect to the base coordinates. Therefore, for our purposes the vanishing order, normally defined in terms of the power of the maximal ideal at the given point in terms of the  $x$  coordinates, will be defined in terms of the degree of vanishing derivatives (in  $x$  coordinates) at the given point. We need to emphasize that we are considering vanishing order of



the smooth maps on the jet bundle only with respect to dependence on the base coordinates.

Given the usual framing  $\partial_i = \frac{\partial}{\partial x_i}$ ,  $1 \leq i \leq m$  for  $T\mathbb{R}^m$ ,  $\partial_{i,x}$  being the frame for  $T_x\mathbb{R}^m$ ; we have an induced framing of  $T\mathcal{J}_{m,n}^k$  given by adjoining to these tangent horizontal vectors the vectors  $\partial_{y_\alpha^j} = \frac{\partial}{\partial y_\alpha^j}$  that are tangent to the fibers of the bundle projection  $\pi_0^k$  at  $x$ , for  $j = 1, \dots, n$  and  $|\alpha| \leq k$ .

The notion of contact is useful in understanding the sharpening of the results here vis a vie the results of Todorov. Given  $x \in \mathbb{R}^m$  and a nonnegative integer  $s$ , we say that  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  has **contact  $s$**  with  $\mathbb{R}^m$  at  $x$  if  $f(x) = 0$  and the graph of  $f$ ,  $\Gamma_f \subset \mathbb{R}^m \times \mathbb{R}^n$  is flat to  $s^{th}$  order at  $(x, 0)$ , that is, if  $T_x^k f = 0$ , ie., if  $j_x^s f$  is the equivalence containing the 0  $s$  jet at  $x$ . We say that  $f, g \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  have  **$s^{th}$  order contact at  $x$**  if  $f - g$  has  $s^{th}$  order contact with  $\mathbb{R}^m$  at  $x$ , the graph of the 0 function, at  $x$ . It should be obvious that this is an equivalence relation and that the set of all  $g \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  that belong to the  $s^{th}$  order contact class of  $f$  is precisely the affine subset with the same  $s^{th}$  order jet as  $f$ .

**2.2.2. Prolonging jet maps and total derivatives.** Let  $\mathbb{R}_m^p$  denote the product bundle with fiber  $\mathbb{R}^p$  and base  $\mathbb{R}^m$ ; if  $p = 1$ , we will denote this bundle by  $\mathbb{R}_m$ . If  $x \in \mathbb{R}^m$ , let  $\mathbb{R}_{m,x}^p$  denote the vector space fiber of  $\mathbb{R}_m^p$  over  $x$ . In the following we will be using **vector bundle maps**, ie., smooth maps of bundles over the same base that preserve fibers and cover the identity map on the base. The symbols of linear differential operators,  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m^p$  are maps of this type. The set of such maps is a  $C^\infty(\mathbb{R}^m, \mathbb{R})$  module and will be denoted by  $C^\infty(\mathcal{J}_{m,1}^r, \mathbb{R}_m^p)$ . If  $\lambda : \mathcal{J}_{m,n}^k \rightarrow \mathbb{R}_m^p$  is such a smooth bundle map, and  $l \in \mathbb{N}$ , then there exists a smooth bundle map  $\lambda^{(l)} : \mathcal{J}_{m,n}^{k+l} \rightarrow \mathcal{J}_{m,p}^l$ , called the  **$l^{th}$ -prolongation of  $\lambda$**  such that the following diagram is commutative

$$(5) \quad \begin{array}{ccc} \mathcal{J}_{m,n}^{k+l} & \xrightarrow{\lambda^{(l)}} & \mathcal{J}_{m,p}^l \\ \pi_k^{k+l} \downarrow & & \downarrow \pi_0^l \\ \mathcal{J}_{m,n}^k & \xrightarrow{\lambda} & \mathbb{R}_m^p \end{array}$$

That is, as  $j_x^s(\lambda \circ j^r f)$  depends only on derivatives up to order  $s$  of  $y \mapsto j_y^r f$  at  $x$ , and so only on the  $r + s$  jet of  $f$  at  $x$ , then the following definition is well defined. If  $f \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ , then

$$(6) \quad \lambda^{(s)}(j_x^{r+s}(f)) = j_x^s(\lambda \circ j^r f).$$

The prolongation of vector fields on  $\mathbb{R}^m$  to vector fields on  $\mathcal{J}_{m,n}^k$  are given by fairly complicated recursion formulas. For treatments of prolongations of vector fields in somewhat different contexts, see Olver [21], p110 or Pommaret, [22], p253. Pommaret gives a remarkably easy derivation of these expressions. We only need the prolongation of coordinate vector fields, ie., total derivatives, and these have far simpler expressions. These give explicit local expressions of prolongations and so allow us to computationally investigate the effect of successive prolongations on maps of jets. For each coordinate tangent field  $\partial_i$  on  $T\mathbb{R}^m$ , for  $1 \leq i \leq m$ , we have an explicit expression for the corresponding lifted local section of  $T\mathcal{J}_{m,n}^k$ , the **total**

**derivative**  $\partial_i^\#$  defined as follows.

$$(7) \quad \partial_i^\# = \partial_i + \sum_{\substack{|\alpha| \leq k \\ 1 \leq j \leq n}} y_{\alpha^i}^j \partial_{y_{\alpha^j}^j}$$

where  $\alpha^i = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_n)$ . Note that  $\partial_i^\#$  depends on coordinates of order  $k + 1$ , ie., for  $\lambda \in C^\infty(\mathcal{J}_{m,n}^k, \mathbb{R}_m^n)$ , we have  $\partial_i^\#(\lambda)$  are the coordinates for a map  $\lambda^{(1)} : \mathcal{J}_{m,n}^{k+1} \rightarrow \mathcal{J}_{m,n}^1$  with respect to the induced jet coordinates. In fact, we have the following.

**Lemma 2.1.** *Suppose that  $\lambda : \mathcal{J}_{m,n}^k \rightarrow \mathbb{R}^n$  is a smooth map. Let  $\lambda^j$  for  $j = 1, \dots, n$  denote the coordinates of  $\lambda$  with respect to the standard coordinate basis for  $\mathbb{R}^n$ . Then the components of  $\lambda^{(1)}$  with respect to the given coordinates are  $\partial_i^\#(\lambda_\alpha^j)$*

*Proof.* One can verify this lemma and the expression (7) using the local version of the definition for prolonging jet maps (6) when  $s = 1$ , eg., see [21], p109; ie.,

$$(8) \quad \partial_i^\#(\lambda)(j_x^{r+1}(f)) = \partial_i(\lambda \circ j^r f)(x)$$

and then applying the chain rule to the right side of (8).  $\square$

**2.3. Differential operators and their prolongations.** In order to align with Todorov's setup we will now restrict the dimension of the range space to be 1. The superscript  $j$  enumerating range space components will no longer appear.

Let  $LPDO_r$  denote the vector space of linear partial differential operators of degree less than or equal to  $r$  with coefficients in  $C^\infty(\mathbb{R}^m, \mathbb{R})$ . Suppose that  $P \in LPDO_r$ . Then there exists a smooth bundle map  $\lambda_P : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m$  called the **total symbol of  $P$**  such that if  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$ , then  $P(f) = \lambda_P \circ j^r f$  as elements of  $C^\infty(\mathbb{R}^m, \mathbb{R})$ . If  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m$  is the symbol of an  $r^{th}$  order differential operator,  $P$ , then the  **$s^{th}$  prolongation  $\lambda^{(s)} : \mathcal{J}_{m,1}^{r+s} \rightarrow \mathbb{R}_m$**  is defined on  $r + s$  jets above. As prolongations of differential operators mapping  $C^\infty(\mathbb{R}^m, \mathbb{R})$  to  $C^\infty(\mathbb{R}^m, \mathbb{R})$  are systems, we will use the notation  $LPDO_{r+s}^s$  for  $r + s$  order linear differential operators on  $C^\infty(\mathbb{R}^m, \mathbb{R})$  to smooth sections of  $\mathcal{J}_{m,1}^s$ . In particular, if  $P \in LPDO_r$ , and  $s \in \mathbb{N}$ , then there is  $P^{(s)} \in LPDO_{r+s}^s$ , called the  **$s^{th}$  prolongation of  $P$** , defined as the differential operator whose symbol is the  $s^{th}$  prolongation of  $\lambda_P$ . We will have more to say about its nature later.

On the nonstandard level, note that if  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$ , then, at every standard  $x$ , ie.,  $*x \in {}^\sigma\mathbb{R}^m$ , and for each  $k \in {}^\sigma\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $*j_{*x}^k(*g) = *(j_x^k g)$ . That is, the internal operator  $*j_{*x}^k|_{{}^\sigma C^\infty(\mathbb{R}^m, \mathbb{R})}$  operating on standard functions is just the transfer of  $j_x^k$ . (At nonstandard points this is not true.) It therefore follows that  $*\lambda \circ *j^k$ , restricted to a standard map,  $*g$  at a standard point  $*x$ , is just the  $*\text{transfer of } \lambda \circ j_x^k g \in \mathbb{R}_{m,x}$ . We shall give a sufficient account of the remainder of the nonstandard material needed after we recall a bit more standard geometry.

### 3. STANDARD GEOMETRY: PROLONGATION AND VANISHING

This section proves the standard results that allow the proof of Corollary 5.3. The idea here is desingularize our total symbol by “lifting” it to a sufficiently high jet level where we can then invoke a version of Todorov's result. In order to do this, we need some sort of correspondence between solutions of  $P$  and those of  $P^{(s)}$ . We also need this procedure to decrease the vanishing order the coefficients of the

$P^{(k)}$  as  $k$  increases. Here is the idea. On the one hand, we can think of a symbol,  $\lambda = \lambda_P$  of an LPDE as a smooth family of linear maps  $x \mapsto \lambda_x$ , and therefore one can think of vanishing order of  $\lambda$  at  $x_0$  as the “flatness” of the graph of this map at the point  $x_0 \in \mathbb{R}^m$ . This relates directly to the Taylor polynomials of the smooth coefficients of  $\lambda$  at  $x_0$ . On the other hand and more abstractly, there is a classic “desingularization” machinery for jet bundle map, eg., symbols of differential operators, that carries solutions to solutions, ie., prolongation. In this section we will relate the intuitive vanishing order to this prolongation method in order to get crude controls between jets in the domain and range of the prolongation of  $\lambda$ , in terms of these singularities. This will be done in this section.

We first need the proper notion of vanishing order of a linear bundle map  $\lambda : \mathcal{J}_{m,1}^k \rightarrow \mathbb{R}$ .

**Definition 3.1.** *Let  $x_0 \in \mathbb{R}^m$ , and  $c \in \mathbb{N}$ . Then we say that  $\lambda$  **vanishes to order (exactly)  $c$  (in  $x$ ) along  $(\pi_0^k)^{-1}(x_0)$** , written  $x_0 \in \mathcal{Z}^c(\lambda)$ , if  $\partial^\alpha(\lambda)(p) = 0$  for all  $\alpha$  with  $|\alpha| \leq c$  and for all  $p \in (\pi_0^k)^{-1}(x_0)$ . and, secondly, there exists some  $\alpha$  and  $\beta$  with  $|\beta| = c + 1$  and  $p_0 \in (\pi_0^k)^{-1}(x_0)$  such that  $\partial^\beta(\lambda)(p_0) \neq 0$ . Note, as always, that  $\partial^\alpha$  denotes the  $\alpha^{th}$  derivative with respect to the  $x_i$  coordinates.*

First note that this is more transparently stated as follows. Writing  $\lambda = \sum_\alpha f_\alpha y_\alpha$  for some smooth  $f_\alpha$ ’s, this condition is equivalent to stating that for each coefficient  $f_\alpha$ ,  $\partial^\beta f_\alpha(x_0) = 0$  for all multiindices  $\beta$  satisfying  $|\beta| \leq k$ ; with the second condition being that there exists a coefficient  $f_\alpha$  and a multiindex  $\beta$  with  $|\beta| = k + 1$  such that  $\partial^\beta f_\alpha(x_0) \neq 0$ . Note also that although the notion of contact is closely related to vanishing order, we will not pursue this connection here. In the next lemma, we don’t need to restrict to jet mappings that are the symbols of elements of  $LPDO_k$ . When we consider how total derivatives change the vanishing order of jet maps, it will be essential to consider the particular form of jet maps that are symbols of elements of  $LPDO_k$ . Below we will be using the following notation. If  $\beta = (\beta_1, \dots, \beta_m)$  is a multiindex of order  $k$ ; ie.,  $|\beta| = \beta_1 + \dots + \beta_m = k$  and  $1 \leq i_0 \leq m$ , then  $\beta_{i_0} = (\beta_1, \dots, \beta_{i_0} + 1, \beta_{i_0+1}, \dots, \beta_m)$ ; so eg.,  $|\beta_{i_0}| = |\beta| + 1$ .

**Lemma 3.1.** *Let  $\lambda \in C^\infty(\mathcal{J}_{m,1}^k, \mathbb{R}_m)$  and suppose that  $p_0 \in \mathcal{J}_{m,1}^k$  is in  $\mathcal{Z}^c(\lambda)$ . Then for some  $i \in \{1, \dots, m\}$ , we have  $p_0 \in \mathcal{Z}^{c-1}(\partial_i(\lambda))$ .*

*Proof.* By hypothesis,  $\partial^\alpha(\lambda)(p_0) = 0$  for  $|\alpha| \leq c$  and there is a multiindex  $\beta$  with  $|\beta| = c + 1$  and an  $\alpha$  such that  $\partial^\beta(\lambda)(p_0) \neq 0$ . Write  $\beta = \alpha_i$  for some multiindex  $\alpha$  with  $|\alpha| = c$  and  $i \in \{1, \dots, m\}$ . That is,  $\partial^\alpha(\partial_i \lambda)(p_0) \neq 0$ . But if  $\alpha$  is a multiindex such that  $|\alpha| \leq c - 1$ , then  $|\alpha_i| \leq c$  and so  $\partial^\alpha(\partial_i \lambda)(p_0) = \partial^{\alpha_i}(\lambda)(p_0) = 0$  by hypothesis. But then by definition  $p_0 \in \mathcal{Z}^{c-1}(\partial_i \lambda)$ , as we wanted to show.  $\square$

**3.1. Lifting solutions.** At this point, it is important to note the explicit form of the jet maps that are symbols of elements of  $LPDO_k$ .

**Lemma 3.2.** *Suppose that  $P \in LPDO_k$ . Then  $\lambda = \lambda_P : \mathcal{J}_{m,1}^k \rightarrow \mathbb{R}_m$  can be written in local coordinates  $(x_i, y_\alpha)$  in the following form  $\lambda = \sum_{|\alpha| \leq k} f_\alpha y_\alpha$  where  $f_\alpha \in C^\infty(\mathbb{R}^m, \mathbb{R})$ .*

*Proof.* This is clear.  $\square$

A remark is in order. Since for each  $x \in \mathbb{R}^m$ ,  $\lambda_P = \lambda_{P,x} : \mathcal{J}_{m,1,x}^r \mathbb{R}_{m,x}$  is linear, then we will consider look of the vanishing order of  $x \mapsto \lambda_x$

**Lemma 3.3.** *Suppose that  $P \in LPDO_r$ ,  $s \in \mathbb{N}$  and  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  solves  $P^{(s)}(f) = 0$ . Then  $f$  solves  $P(f) = 0$ .*

*Proof.* This is an easy unfolding of the definitions. Operationally,  $P^{(s)}$  is defined on  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  as follows.  $P^{(s)}(f) = j^s \circ P(f)$ . But then  $P^{(s)}(f) = 0$  implies that  $j^s \circ P(f) = 0$ . But for  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$ ,  $j^s(g) = 0$  as a section of  $\mathcal{J}_{m,1}^s$  if and only if  $g$  is identically 0, eg., letting  $g = P(f)$ , we get  $P(f) = 0$ .  $\square$

### 3.2. Lifting zero sets.

#### 3.2.1. Prolongation of linear symbols.

**Remark.** If  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m$  is the symbol of  $P \in LPDO_r$ , then we **will instead write  $\mathcal{Z}^c(\lambda)$  for  $\pi^r(\mathcal{Z}^c(\lambda))$** . In particular,  $\mathcal{Z}^c(\lambda)$  will now be considered as a subset of the base space,  $\mathbb{R}^m$ . Note that since  $\lambda$  is linear on the fibers, with particular form as noted in Lemma 3.2, then to say that  $x \mapsto \lambda_x$  vanishes to order  $c$  at  $x_0$  is the same as our fiber condition as this says that the coefficient  $f_\alpha$  of our generic jet derivative  $y_\alpha$  vanishes to order  $c$  at  $x_0$ . So, in the linear case, we have the following definition.

**Definition 3.2.** *Suppose that  $\lambda$  is the symbol of  $P \in LPDO_r$ . Let  $\mathcal{Z}^c(\lambda)$  denote the  $x \in \mathbb{R}^m$  where all of the coefficients of  $\lambda$  vanish to order  $c$  and at least one does not vanish to order  $c+1$ . If  $\lambda$  is such a linear jet bundle map, let  $\mathcal{Z}_\lambda$  denote those  $x \in \mathbb{R}^m$  where  $\lambda_x$  is the zero linear map. Given this, it should be obvious that the conclusion of Lemma 3.1 holds with  $\pi^r(\mathcal{Z}^c(\lambda))$  in place of  $\mathcal{Z}^c(\lambda)$ .*

With this development, we have the following initiating lemma.

**Lemma 3.4.** *Suppose that  $P \in LPDO_r$  with  $\lambda \in C^\infty(\mathcal{J}_{m,1}^r, \mathbb{R}_m)$  the symbol of  $P$ . Let  $c \in \mathbb{N}$  and  $x_0 \in \mathcal{Z}^c(\lambda)$ . Then  $x_0 \in \mathcal{Z}^{c-1}(\lambda^{(1)})$ . In particular, if  $x_0 \in \mathcal{Z}^1(\lambda)$ , then  $\lambda_{x_0}^{(1)} \neq 0$ .*

*Proof.* So we have that  $x_0$  satisfies  $\partial^\alpha \lambda(x_0) = 0$  if  $|\alpha| \leq c$ , and there exists multiindex  $\beta$ , with  $|\beta| = c+1$  such that  $\partial^\beta \lambda(x_0) \neq 0$ . By Lemma (2.1) we only need to verify that for all  $i, \alpha$  with  $|\alpha| \leq c-1$ ,  $\partial^\alpha (\partial_i^\# \lambda)(x_0) = 0$  and there exists  $i_0, \tilde{\alpha}$  with  $|\tilde{\alpha}| = c$  such that  $\partial^{\tilde{\alpha}} (\partial_{i_0}^\# \lambda)(x_0) \neq 0$ . To this end, suppose that the following statement is valid. Let  $d \in \mathbb{N}$  and suppose that for all  $\alpha$  with  $|\alpha| \leq d$ , we have that  $\partial^\alpha \lambda(p_0) = 0$ . Then for arbitrary given  $i_0$  and  $\tilde{\alpha}$  with  $|\tilde{\alpha}| = d$ ,  $\partial^{\tilde{\alpha}} (\partial_{i_0}^\# \lambda)(p_0) = 0$  if and only if  $\partial^{\tilde{\alpha}} (\partial_{i_0} \lambda)(p_0) = 0$ . Then one can see that from this statement and Lemma 3.1 the proof of our lemma follows immediately; so it suffices to prove the above statement. First of all, we need only to verify this statement for  $\lambda = \lambda_P$ , for  $P = \sum_{|\alpha| \leq r} f_\alpha \partial^\alpha$  where  $f_\alpha \in C^\infty(\mathbb{R}^m, \mathbb{R})$  for all  $\alpha$ ; that is, if  $\lambda = \sum_{|\alpha| \leq r} f_\alpha y_\alpha$ . In the following calculations we will leave out evaluation at  $x_0$ ; it is implicit. So

$$\begin{aligned}
 \partial_i^\#(\lambda) &= (\partial_i + \sum_{|\alpha| \leq r} y_{\alpha_i} \partial_{y_\alpha}) \left( \sum_{|\beta| \leq r} f_\beta y_\beta \right) \\
 &= \sum_{|\beta| \leq r} \partial_i(f_\beta) y_\beta + \sum_{|\alpha| \leq r} f_\alpha y_{\alpha_i}.
 \end{aligned}
 \tag{9}$$

That is

$$(10) \quad \partial_i^\#(\lambda) = \sum_{|\alpha| \leq r} ((\partial_i f_\alpha) y_\alpha + f_\alpha y_{\alpha_i}).$$

so that taking  $\partial^{\tilde{\alpha}}$ , for  $|\tilde{\alpha}| \leq d$ , of both sides and interchanging derivatives, we get

$$(11) \quad \partial_i^\#(\partial^{\tilde{\alpha}} \lambda) = \sum_{|\beta| \leq r} \partial_i(\partial^{\tilde{\alpha}} f_\beta) y_\beta + \partial^{\tilde{\alpha}}(f_\beta) y_{\beta_i}.$$

But by hypothesis, the second term, on the right side of (11) is 0 and the truth of the statement follows.  $\square$

**3.2.2. Higher prolongations; coordinate calculations.** We want to prove a higher order prolongation version of the previous lemma. We need some preliminaries before we state the lemma. Suppose that  $\alpha$  is a multiindex, let  $\partial_\alpha^\#$  denote the  $\alpha^{th}$  iteration of the coordinate total derivatives; ie., if  $\alpha = (\alpha_1, \dots, \alpha_m)$  is a multiindex of order  $k = \alpha_1 + \dots + \alpha_m$ , then  $\partial_\alpha^\# = (\partial_1^\#)^{\alpha_1} \circ \dots \circ (\partial_m^\#)^{\alpha_m}$ , it being understood that if  $\alpha_j = 0$ , then the corresponding  $j^{th}$  factor is missing. As defined,  $\partial_\alpha^\#$  sends functions on  $\mathcal{J}_{m,1}^r$  to functions on  $\mathcal{J}_{m,1}^{r+k}$ . Next, note that if  $\lambda = \sum_\alpha f_\alpha y_\alpha$  is the symbol of  $P \in LPDO_r$ , then  $\lambda^{(k)}$  is the symbol of the  $r + k^{th}$  order operator  $P^{(s)}$ . As such, using the induced coordinates  $y_\alpha$ ,  $|\alpha| \leq k$  on the range,  $\mathcal{J}_{m,1}^k$ , we can write  $\lambda_x^{(k)}$  as  $((\partial_\alpha^\# \lambda)_x)_{|\alpha| \leq k}$ . That is,  $\lambda^{(k)}$  is given by the family of linear maps

$$(12) \quad x \mapsto ((\partial_\alpha^\# \lambda)_x)_\alpha : \mathcal{J}_{m,1,x}^{r+k} \rightarrow \mathcal{J}_{m,1,x}^k$$

Note then that this family of maps vanishes to order  $c$  at  $x_0$  precisely when for each  $\alpha$ ,  $|\alpha| \leq r + k$  the component  $x \mapsto \partial_\alpha^\#(\lambda)_x$  vanishes to order  $c$ . Given this, we have the following extension of the previous lemma to general prolongations.

**Lemma 3.5.** *Suppose that the bundle map  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m$  is the symbol of  $P \in LPDO_r$  and suppose that  $x_0 \in \mathcal{Z}^c(\lambda)$ . Then  $\lambda_{x_0}^{(c+1)}$  is a nonzero linear map.*

*Proof.* By the remarks before the lemma, it suffices to prove that  $\lambda_{x_0}^{(c+1)}$  has a nonzero component at  $x_0$ ; that is, for some  $\gamma$  with  $|\gamma| \leq c + 1$ ,  $\partial_\gamma^\#(\lambda)_{x_0}$  is a nonzero linear map. Note also that if  $\alpha, \beta$  are distinct multiindices, then  $\partial_\alpha^\# \partial_\beta^\# = 0$ . First note that if  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$  and  $\alpha, \gamma$  are appropriate multiindices, then

$$(13) \quad \partial_\gamma^\#(gy_\alpha) = \sum_{\epsilon + \rho = \gamma} \partial_\epsilon g \cdot y_{\alpha + \rho}$$

Suppose that  $g$  vanishes to order (exactly) at  $x_0$ ; so that there is an index  $i_0$  and multiindex  $\tilde{\beta}$  with  $|\tilde{\beta}| = c$ , such that  $\partial_\alpha g(x_0) = 0$  for  $|\alpha| \leq c$  and  $\partial_{\tilde{\beta}_{i_0}} g(x_0) \neq 0$ . Then  $\partial_{\tilde{\beta}_{i_0}}^\#(gy_\alpha) = \partial_{\tilde{\beta}_{i_0}} g \cdot y_\alpha$ . This follows upon inspection:  $\tilde{\beta}_{i_0}$  is the only multiindex of length  $c + 1$  occurring in the sum. All other multiindices occurring are of length  $\leq c$  and so these terms are zero by the hypotheses on  $g$ .

So now consider a general symbol  $\lambda = \sum_{|\alpha| \leq r} f_\alpha y_\alpha$  and suppose, by hypothesis that  $x_0 \in \mathcal{Z}^c(\lambda)$ . So there exists a multiindex  $\alpha_0$  with  $|\alpha_0| \leq r$ , a multiindex  $\tilde{\beta}$  of order  $c$  and an index  $i_0 \in \{1, \dots, m\}$  such that  $\partial_{\tilde{\beta}_{i_0}} f_{\alpha_0}(x_0) \neq 0$ . We will show that

the  $\bar{\beta}_{i_0}$  component of  $\lambda_{x_0}^{(c+1)}$  is nonzero; ie., that  $\partial_{\bar{\beta}_{i_0}}^\#(\lambda)_{x_0}$  is a nonzero linear map. Now

$$\begin{aligned}
 \partial_{\bar{\beta}_{i_0}}^\#(\lambda)_{x_0} &= \sum_{|\alpha| \leq r} \partial_{\bar{\beta}_{i_0}}^\#(f_\alpha y_\alpha)_{x_0} \\
 &= \sum_{|\alpha| \leq r} \sum_{\gamma \leq \bar{\beta}_{i_0}} \partial_\gamma(f_\alpha)(x_0)(y_{\alpha+(\bar{\beta}_{i_0}-\gamma)})_{x_0} \\
 &= \sum_{|\alpha| \leq r} \sum_{\substack{\gamma \leq \bar{\beta}_{i_0} \\ |\gamma|=c+1}} \partial_\gamma(f_\alpha)(x_0)(y_{\alpha+(\bar{\beta}_{i_0}-\gamma)})_{x_0} \\
 (14) \qquad &= \sum_{|\alpha| \leq r} \partial_{\bar{\beta}_{i_0}} f_\alpha(x_0) y_\alpha|_{x_0}.
 \end{aligned}$$

But the linear forms  $y_\alpha|_{x_0}$  are linearly independent and the coefficient of  $y_{\alpha_0}|_{x_0}$  is nonzero by hypothesis, hence the conclusion follows.  $\square$

**Corollary 3.1.** *Suppose that  $P \in LPDO_r$  with  $\lambda \in C^\infty(\mathcal{J}_{m,1}^r, \mathbb{R}_m)$  the symbol of  $P$ . Suppose that  $c_0 = \sup\{c \in \mathbb{N} : \mathcal{Z}_\lambda^c \text{ is nonempty}\}$  is finite. Then  $\mathcal{Z}(\lambda^{(c_0+1)})$  is empty. That is, for each  $x \in \mathbb{R}^m$ ,  $\text{rank}(\lambda_x^{(c_0)}) \geq 1$ .*

*Proof.* This is an immediate consequence of the above lemma.  $\square$

#### 4. STANDARD GEOMETRY: PROLONGATION AND RANK

In section 3, we were concerned with the vanishing order of the (total) symbol of an element of  $LPDO_r$ . Our proofs involved calculations with the induced local coordinate formulation of prolongations of jet bundle maps. In this section, we will prove the standard results needed in the proof of Corollary 5.3. Here, we are a bit more traditionally concerned with the prolongation effects of regularity hypotheses on the principal symbol of an element of  $LPDO_r$ . Our constructions will instead be in the tradition of diagram chasing through commutative diagrams of jet bundles.

Here we will show that if the principal symbol  $\underline{\lambda} : \mathcal{J}_{m,1,x}^r \rightarrow \mathbb{R}_m$  is nonvanishing, then for each  $k \in \mathbb{N}$ ,  $\lambda^{(k)} : \mathcal{J}_{m,1,x}^{r+k} \rightarrow \mathcal{J}_{m,1,x}^k$  is maximal rank. We will use this fact in constructing solutions for  $P(f) = g$ . We will first look at a coordinate argument that seems to indicate this. We will follow this with a simple version with a proof of this fact using a typical jet bundle argument.

Suppose that  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m$  is the symbol of  $P \in LPDO_r$ . Let's begin by looking at first order prolongations. Suppose that  $x_0 \in \mathbb{R}^m$  and that  $\lambda_{x_0} : \mathcal{J}_{m,1,x_0}^r \rightarrow \mathbb{R}_{x_0}$  is nonzero in the sense that some coefficient  $a_{\bar{\alpha}}$ , for  $|\bar{\alpha}| = r$  is nonzero at  $x_0$ . Then we will verify the first prolongation of  $\lambda$ ,  $\lambda_{x_0}^{(1)} : \mathcal{J}_{m,1,x_0}^{r+1} \rightarrow \mathcal{J}_{m,1,x_0}^1$  has "top order part" of maximal rank. So we need to show that the rank of the "top order part" of  $\lambda_{x_0}^{(1)} : \mathcal{J}_{m,1,x_0}^{r+1} \rightarrow \mathcal{J}_{m,1,x_0}^1$  is  $m$  as this is the dimension of the fiber of  $\mathcal{J}_{m,1}^1$ . Write  $\lambda_x = \sum_{|\alpha| \leq r} a_\alpha(x) y_\alpha$  and, in coordinates,

$$(15) \qquad \lambda_{x_0}^{(1)} = (\lambda_{x_0}, \partial_1^\# \lambda_{x_0}, \dots, \partial_m^\# \lambda_{x_0})$$

where, as before for each  $i$ ,

$$(16) \quad \partial_i^\# \lambda_{x_0} = \sum_{|\alpha| \leq r} (\partial_i a_\alpha(x_0) y_\alpha + a_\alpha(x_0) y_{\alpha_i}).$$

where by assumption  $a_{\tilde{\alpha}}(x_0) \neq 0$  for some  $\tilde{\alpha}$  with  $|\tilde{\alpha}| = r$ . But then the linear forms  $a_{\tilde{\alpha}}(x_0) y_{\tilde{\alpha}_i}$  for  $i = 1, \dots, m$  are linearly independent. Therefore, the linear forms  $\partial_i^\# \lambda_{x_0}$ , by their above expressions (16), are also linear independent. Hence, their image spans the fiber of  $\mathcal{J}_{m,1}^1 \rightarrow \mathbb{R}^m$  over  $x_0$ .

Given that  $\lambda$  is the (total) symbol of  $P \in LPDO_r$ ; the “top order part” of  $\lambda$ ,  $\underline{\lambda}$  is the principal symbol of  $P$ . We need a little more jet stuff to properly define the principal symbol and proceed with the general statement for arbitrary prolongations. The set of  $j_x^{k+1} f$  of  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  that vanish to  $k^{th}$  order at  $x$  have a canonical  $\mathbb{R}$  vector space identification with  $\mathcal{S}_{k+1} T_x^*$ , the  $k+1^{st}$  symmetric power of  $T_x^*$ , the cotangent space to  $\mathbb{R}^m$  at  $x$ . (See Pommaret, [22], p47, 48.) In fact, for every  $x \in \mathbb{R}^m$  and  $k \in \mathbb{N}$  we have a canonical injection of vector spaces,  $\mathcal{S}_k T_x^* \xrightarrow{i_k} \mathcal{J}_{m,1,x}^k$  which is a canonical isomorphism when  $k = 1$ , ie.,  $T_x^* \cong \mathcal{J}_{m,1,x}^1$ . This injection embeds in an exact sequence:

$$(17) \quad 0 \rightarrow \mathcal{S}_{k+1} T_x^* \xrightarrow{i_{k+1}} \mathcal{J}_{m,1,x}^{k+1} \xrightarrow{\pi_{k+1}^{k+1}} \mathcal{J}_{m,1,x}^k \rightarrow 0.$$

In fact this and much of the following holds in far greater generality; eg., giving exact sequences of jets of bundles over any paracompact smooth manifold. Next, when  $k = r$  the order of our operator, note that expression (16) when evaluated on  $j_x^{r+1} f \in \mathcal{S}_{k+1} T_x^*$  gives

$$(18) \quad \partial_i^\# \lambda_x(j_x^{k+1} f) = \sum_{|\alpha| \leq r} a_\alpha(x) y_{\alpha_i} (j_x^{k+1} f),$$

the other terms being zero as  $j_x^k f = 0$ . Now for each  $k = 0, 1, 2, \dots$ , the **principal symbol of  $P^{(k)}$** , denoted  $\underline{\lambda}^{(k)} : \mathcal{S}_{r+k} T_x^* \rightarrow \mathcal{S}_k T_x^*$  is the  $\mathbb{R}$  linear map induced by the restriction to  $\mathcal{S}_k T_x^*$  of  $\lambda^{(k)}$ . As the principal symbol  $\underline{\lambda} : \mathcal{S}_r T_x^* \rightarrow \mathbb{R}_{m,x}$  can be represented by  $\sum_{|\alpha|=r} a_\alpha y_\alpha$ , then  $\underline{\lambda}^{(1)} : \mathcal{S}_{r+1} T_x^* \rightarrow T_x^*$  can be written  $\sum_i \sum_{|\alpha|=r} a_\alpha y_{\alpha_i} \otimes dx_i$ . See the discussion below. Also note that the linear maps  $y_\alpha|_{\mathcal{S}_k T_x^*}$  decomposes as the  $\alpha$  symmetric product of the coordinate cotangent vectors and we have the canonical  $\mathbb{R}$  vector space identification  $\mathcal{S}_{r+1} T_x^* \cong \mathcal{S}_r T_x^* \otimes T_x^*$ . With these preliminaries we can prove the following.

**Lemma 4.1.** *Suppose that for  $x \in \mathbb{R}^m$ ,  $\underline{\lambda}_x : \mathcal{J}_{m,1,x}^r \rightarrow \mathbb{R}_{m,x}$  is nonzero, ie. a surjection. Then  $\lambda^{(k)}$  is a surjection for all  $k \in \mathbb{N}$ .*

*Proof.* The remark above allows us to decompose the expression for  $\underline{\lambda}^{(1)}$  as  $1_{T_x^*} \otimes \underline{\lambda}$ . Actually this holds at all levels. (SEE POMMARRET, [22] p193) That is,

$$(19) \quad \underline{\lambda}_x^{(k)} = \underline{\lambda}_x \otimes 1|_{\mathcal{S}_k T_x^*}.$$

But note that in the category of finite dimensional  $\mathbb{R}$  vector spaces, the tensor product of surjections is a surjection, hence by hypothesis  $\underline{\lambda}^{(k)}$  is a surjection for each  $k \in \mathbb{N}$ .

We will prove, by induction on  $k$ , the order of prolongation, that  $\lambda^{(k)}$  is a surjection. The result holds for  $k = 0$ . Suppose that it holds for some  $k \geq 0$ . We will prove that it holds for  $k + 1$ . By the inductive hypothesis, the remark on  $\underline{\lambda}^{(k)}$  directly above and general facts on jets, we have a commutative diagram of exact

sequences of linear maps over  $x$

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
 & \mathcal{S}_{r+k+1}T_x^* & \xrightarrow{\underline{\lambda}^{(k+1)}} & \mathcal{S}_{k+1}T_x^* & \longrightarrow 0 \\
 & \downarrow 1_{r+k+1} & & \downarrow 1_{k+1} & \\
 (20) & \mathcal{J}_{m,1,x}^{r+k+1} & \xrightarrow{\lambda^{(k+1)}} & \mathcal{J}_{m,1,x}^{k+1} & \\
 & \downarrow \pi_{r+k}^{r+k+1} & & \downarrow \pi_k^{k+1} & \\
 & \mathcal{J}_{m,1,x}^{r+k} & \xrightarrow{\lambda^{(k)}} & \mathcal{J}_{m,1,x}^k & \longrightarrow 0 \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 & 
 \end{array}$$

and we wish to prove that the middle row is a surjection. Suppose that  $\eta_{k+1} \in \mathcal{J}_{m,1,x}^{k+1}$ . We will find  $\zeta \in \mathcal{J}_{m,1,x}^{r+k+1}$  such that  $\lambda^{(k+1)}(\zeta) = \eta_{k+1}$ . The proof will be a typical 'diagram chase'. Let  $\eta_k = \pi_k^{k+1}(\eta_{k+1})$ . Then, by hypothesis, there exist  $\eta_{r+k} \in \mathcal{J}_{m,1,x}^k$  such that  $\lambda^{(k)}(\eta_{r+k}) = \eta_k$ . Let  $\eta_{r+k+1} \in (\pi_{r+k}^{r+k+1})^{-1}(\eta_{r+k})$ . So commutativity of the lower square implies that

$$(21) \quad \pi_k^{k+1} \circ \lambda^{(k+1)}(\eta_{r+k+1}) = \lambda^{(k)} \circ \pi_{r+k}^{r+k+1}(\eta_{r+k+1}) = \eta_k = \pi_k^{k+1}(\eta_{k+1}).$$

That is, 1)  $\pi_k^{k+1}(\lambda^{(k+1)}(\eta_{r+k+1}) - \eta_{k+1}) = 0$ , and so by exactness of the right sequence, there is  $\sigma_{k+1} \in \mathcal{S}_{k+1}T_x^*$  such that

$$(22) \quad i_{k+1}(\sigma_{k+1}) = \lambda^{(k+1)}(\eta_{r+k+1}) - \eta_{k+1}$$

But  $\underline{\lambda}^{(k+1)}$  is surjective, ie., there exists  $\sigma_{r+k+1} \in \mathcal{S}_{r+k+1}T_x^*$  such that  $\underline{\lambda}^{(k+1)}(\sigma_{r+k+1}) = \sigma_{k+1}$ . So by this and commutativity of the top square, we have

$$(23) \quad \lambda^{(k+1)} \circ i_{r+k+1}(\sigma_{r+k+1}) = i_{k+1} \circ \underline{\lambda}^{(k+1)}(\sigma_{r+k+1}) = i_{k+1}(\sigma_{k+1}).$$

Combining the equivalences in (22) and (23), we get

$$(24) \quad \lambda^{(k+1)}(\eta_{r+k+1} - i_{r+k+1}(\sigma_{r+k+1})) = \eta_{k+1}.$$

That is,  $\zeta \doteq \eta_{r+k+1} - i_{r+k+1}(\sigma_{r+k+1})$  is the element of  $\mathcal{J}_{m,1,x}^{r+k+1}$  we are looking for.  $\square$

Note that we did not use the full strength of our setting; we did not use the exactness of the left vertical sequence.

**Corollary 4.1.** *Suppose that  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m$  is such that  $\lambda_{x_0} : \mathcal{J}_{m,1,x}^r \rightarrow \mathbb{R}_{x_0}$  is nonzero as in the previous lemma. Then if  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$ ,  $j_{x_0}^k g \in \text{Im}(\lambda_{x_0}^{(k)})$ , for every  $k \in \mathbb{N}_0$ .*

*Proof.* This is an immediate consequence of the previous lemma.  $\square$

This corollary will allow us to extend Todorov's pointwise equality to an infinite jetwise equality in the next section.



## 5. THE MAIN LINEAR THEOREM

Before we begin the transfer of our results to the nonstandard world, we need in place a bit more of the framework for the infinite jet results in this section. First, let  $N_s$  denote the finite set of multiindices of length  $m$  and weight less than or equal to  $s$ ; ie., indexing the fiber jet coordinates for  $\mathcal{J}_{m,1}^r$ . Let  $\overline{N_s} \subset N_s$  denote the subset of multiindices of length equal to  $\alpha$ . Given the notational material, we now examine how the lifting works. Todorov proved a result crudely stated as follows: given  $g$ , there exists nonstandard  $f$  such that  $P(f)(x) = g(x)$  at each standard  $x$ . Our intention is to prove that such  $f$  exists such that  $j_x^s(P(f)) = j_x^s g$  for all standard  $x$  and all  $s \in {}^\sigma\mathbb{N}$ . This will be a consequence of the material in the previous section, the transfer of the Borel lemma and a bit more standard preliminaries. The mapping  $\lambda^{(s)}$  can be seen as the intermediary of  $j^s(P(f))$  as follows. If  $s \in \mathbb{N}$ , and  $|\alpha| \leq s$ , we have that

$$(25) \quad j_{x_0}^s(P(f)) = P^{(s)}(f)(x_0) = \lambda_{x_0}^{(s)}(j_{x_0}^{r+s}(f)) = j_{x_0}^s(\lambda \circ j^r f) = j_{x_0}^s g.$$

We can therefore get a good estimate on the size of the range the successive prolongations of the range of  $P$  at  $x_0$  by watching the mapping properties of  $\lambda^{(s)}$ .

We will denote by  $\lambda^{(\infty)}$  the infinite prolongation of  $\lambda$  given by

$$(26) \quad j_x^{r,\infty} f \mapsto j_x^\infty(\lambda \circ j^r f) : \mathcal{J}_{m,1,x}^{(r,\infty)} \rightarrow \mathcal{J}_{m,1,x}^{(\infty)}$$

where  $j_x^{r,\infty} f = (j_x^r f, j_x^{r+1} f, j_x^{r+2} f, \dots)$ , ie.,  $\lambda^{(\infty)}$  being the map whose components are already defined.

In this section and the next the transfer of the Borel Lemma will be used. Here is a statement of the version we will use.

**Lemma 5.1** (Borel Lemma). *Let  $x \in \mathbb{R}^m$  and suppose that  $\phi \in \mathcal{J}_{m,1,x}^\infty$ . Then there exists  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  $\phi = j_x^\infty f$ .*

Note, implicit in this result is the fact that this determination depends only on the germ of  $f$  at  $x$ .

**5.1. Transfer of jet preliminaries.** To prove the main theorem we need to transfer the above jet formulation to the internal arena inserting the homogeneous version of Todorov's result into a jet level high enough so that the symbol has the correct form.

If  ${}^\sigma LPDO_r$  denotes those elements  $P$  of  ${}^*LPDO_r$  whose coefficients are standard elements of  $C^\infty(\mathbb{R}^m, \mathbb{R})$ , then these correspond to symbols  $\lambda_P \in {}^\sigma C^\infty(\mathcal{J}_{m,1}^r, \mathbb{R})$ . Therefore, a special case of the  ${}^*$ transfer of Corollary 3.1 is the following statement.

**Corollary 5.1.** *Let  $r \in {}^\sigma\mathbb{N}$ ,  $D_a = \{x \in \mathbb{R}^m : |x| \leq a\}$  and  ${}^*P \in {}^\sigma LPDO_r$  with  $\lambda$  the symbol of  $P$ . Suppose that  $\max\{c \in {}^\sigma\mathbb{N} : {}^*\mathcal{Z}^c(\lambda) \cap {}^*D_a \text{ is nonempty}\}$  is bounded in  ${}^\sigma\mathbb{N}$  independent of  $a \in \mathbb{N}$ . Then there exists  $s \in {}^\sigma\mathbb{N}$  such that if  $\lambda'$  is the symbol of  $P^{(s)}$ , then  $\mathcal{Z}_{\lambda'} \cap {}^*\mathbb{R}_{nes}^m$  is empty.*

*Proof.* In  ${}^*$ transferring Corollary 3.1, we need only note the following things for this corollary to follow. First of all, we  ${}^*$ transfer this corollary, for the situation where  $\mathcal{Z}(\lambda_P) \subset D_a$  for a given  $0 < a \in \mathbb{N}$  noting that  $\cup_{a>0} {}^*D_a = {}^*\mathbb{R}_{nes}^m$  and that the hypothesis implies that there exists  $a_0 \in \mathbb{N}$  such that  $m(a) \doteq \max\{c \in {}^\sigma\mathbb{N} : {}^*\mathcal{Z}^c(\lambda) \cap {}^*D_a \text{ is nonempty}\}$  satisfies  $m(a) \leq m(a_0)$  for all  $a \in \mathbb{N}$ .  $\square$

**Remark.** Suppose that  $\lambda_P \in C^\infty(\mathcal{J}_{m,1}^r, \mathbb{R})$  is such that for every bounded  $B \subset \mathbb{R}^m$ ,  $\mathcal{Z}(\lambda_P) \cap B$  has no accumulation points. Then  $^*\lambda_P$  can't have the property that  $\lambda_P$  vanishes to infinite, but hyperfinite order at some point in  $^*\mathbb{R}_{nes}^m$ . It therefore follows that we can't use  $^*$ transfer to generalize this result, in the given context, to points where  $\lambda_P$  vanishes to infinite, hyperfinite order.

In order to proceed we need a particular type of nonstandard partition of unity construction. For  $0 < c \in ^*\mathbb{R}^m$  and  $y \in ^*\mathbb{R}^m$ , let  $D_c(y)$  denote the disk centered at  $y$  with radius  $c$ .

**Lemma 5.2** (*\*Weak partition of unity*). *Suppose that for every  $x \in \mathbb{R}^m$ , we have  $f^x \in ^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ . Then there exists  $f \in ^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  and  $0 < \delta \sim 0$  such that for each  $x \in \mathbb{R}^m$ ,  $f|_{D_\delta(x)} = f^x|_{D_\delta(x)}$ .*

*Proof.* First of all, sufficient saturation implies that the (external) map  $\sigma\mathbb{R}^m \rightarrow ^*C^\infty(\mathbb{R}^m, \mathbb{R}^n) : x \mapsto f^x$  extends to an internal map  $\mathcal{I} : ^*\mathbb{R}^m \rightarrow ^*C^\infty(\mathbb{R}^m, \mathbb{R}^n) : l \in \mathcal{I} \mapsto f^l$ ; see Theorem 2.1. Let  $\mathcal{L} \subset ^*\mathbb{R}^m$  be a  $^*$ finite subset such that  $\sigma\mathbb{R}^m \subset \mathcal{L}$ . Choose  $0 < \delta \in ^*\mathbb{R}$  such that  $\delta < \frac{1}{10} \min\{|l - l'| : l, l' \in \mathcal{L}, l \neq l'\}$ . By the  $^*$ transfer of a variation on a weak form of the partition of unity construction, there exists  $\psi_l \in ^*C^\infty(\mathbb{R}^m, \mathbb{R})$  for each  $l \in \mathcal{L}$  such that  $\sum_{l \in \mathcal{L}} \psi_l(x) = 1$  for each  $x \in ^*\mathbb{R}^m$  and for each  $l \in \mathcal{L}$ ,  $\psi_l|_{D_\delta(l)} \equiv 1$ . (As the  $^*$ cardinality of  $\mathcal{L}$  is  $^*$ finite, we don't have to worry about  $^*$ local finiteness of the sum of the  $\psi_l$ 's.) Then the function  $f \doteq \sum_{l \in \mathcal{L}} \psi_l f^l$  has the properties we need.  $\square$

**Remark.** In a follow up paper, a numerically controlled version of this lemma (and the corresponding one in the nonlinear section) will allow proof of most of the existence results in this paper within the category of Colombeau-Todorov algebras.

Let  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathcal{J}_{m,1}^0$  denote the symbol of a  $P \in LPDO_r$ .

**Definition 5.1.** Let  $finsupp(P)$  or  $finsupp(\lambda)$  denote the subset of  $\mathbb{R}^m$  given by  $\cup\{\mathcal{Z}^c(\lambda) : c = 0, 1, 2, \dots\}$ . For each  $x \in \mathbb{R}^m$  and  $k \in \mathbb{N}_0$ , let  $\mathcal{J}_{\lambda,x}^k$  denote the subspace of  $\mathcal{J}_{m,1}^k$  given by  $\lambda^{(k)}(\mathcal{J}_{m,1,x}^{r+k})$ . We write  $g \neq 0(\lambda, x)$ , if  $j_x^k g \neq 0$  for some  $k \in \mathbb{N}$ . Let  $\mathcal{V}_x^m < C^\infty(\mathbb{R}^m, \mathbb{R})$  denote the ideal of  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  $j_x^\infty f = 0$ .

**Lemma 5.3.** *If  $x \in finsupp(\lambda)$ , then there exists  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  $j_x^{k_0} g \neq 0$  for some  $k_0 \in \mathbb{N}$  and  $j_x^k g \in \mathcal{J}_{\lambda,x}^k$  for every integer  $k \geq 0$ .*

*Proof.* Given Corollary 3.1 the assertion amounts to specifying that the derivatives at each level must lie in a given set and hence is an easy consequence of the Borel lemma.  $\square$

**Definition 5.2.** Let  $\mathfrak{J}_{\lambda,x} = \{g \in C^\infty(\mathbb{R}^m, \mathbb{R}) : j_x^k g \in \mathcal{J}_{\lambda,x}^k \text{ for all } k \in \mathbb{N}_0\}$ .

Note, of course, that  $\mathcal{V}_x^m < \mathfrak{J}_{\lambda,x}$ . So by the above lemma,  $\mathfrak{J}_{\lambda,x}$  is infinite dimensional. Therefore,  $^*\mathfrak{J}_{*\lambda,*x}$  is a  $^*$ infinite dimensional  $^*\mathbb{R}$  subspace of  $^*C^\infty(\mathbb{R}^m, \mathbb{R})$ . In the nonstandard world, we have the following analogous definition.

**Definition 5.3.** Let

$$(27) \quad ^*\mathfrak{J}_{\lambda,x} = \{g \in ^*C^\infty(\mathbb{R}^m, \mathbb{R}) : ^*j_{*x}^k g \in ^*\mathcal{J}_{*\lambda,*x}^k \text{ for all } k \in ^*\mathbb{N}_0\}.$$

Note that  $^*\mathfrak{J}_{\lambda,x}$  is an external  $^*\mathbb{R}$  vector space and  $^*\mathfrak{J}_{\lambda,x} \subset ^*\mathfrak{J}_{\lambda,x} \subset ^*C^\infty(\mathbb{R}^m, \mathbb{R})$ . In particular,  $^*\mathfrak{J}_{\lambda,x}$  is infinite dimensional. Note that its  $^*$ dimensionality is not well defined. We have one more definition.

**Definition 5.4.** Let  ${}^\sigma\mathcal{J}_\lambda$  denote the set of  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that for all  ${}^*x \in {}^\sigma\mathbb{R}^m$ ,  $g \in {}^\sigma\mathcal{J}_{\lambda, x}$ .

**Lemma 5.4.** Suppose that  ${}^\sigma\mathcal{J}_{*\lambda, *x} \neq 0$  for some  $x \in \mathbb{R}^m$ . Then  ${}^\sigma\mathcal{J}_\lambda \neq 0$ .

*Proof.* For each  $x \in \mathbb{R}^m$ , choose  $f^x \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  with  $f^x \in {}^\sigma\mathcal{J}_{*\lambda, *x}$ , such that for some  $x$ ,  $f^x \neq 0(\lambda, x)$ . By Lemma 5.2, there exists  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  and  $0 < \delta \sim 0$  such that  $f|_{D_\delta(x)} = f^x|_{D_\delta(x)}$  for each  $x \in \mathbb{R}^m$ . But then, for each  $x \in \mathbb{R}^m$  and each  $k \in \mathbb{N}_0$ ,  ${}^*j_{*x}^k f = {}^*j_{*x}^k f^x \in {}^\sigma\mathcal{J}_{*\lambda, *x}^k$ . That is  $f \in {}^\sigma\mathcal{J}_\lambda$ , and  $f \neq 0(\lambda, x)$  for some  $x$ .  $\square$

**Lemma 5.5.** Let  $x \in \mathbb{R}^m$ , and  $g \in \mathcal{J}_{\lambda, x}$ . Then there exists  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  $\lambda_x^{(\infty)}(j_x^{r, \infty} f) = j_x^\infty g$ .

*Proof.* First of all, for every  $k \in \mathbb{N}_0$ , there exists  $\gamma_k \in \mathcal{J}_{m, 1}^{r+k}$  such that  $\lambda^{(k)}(\gamma_k) = j_x^k g$ . This just follows from the definition of  $\mathcal{J}_{\lambda, x}$ . Since this holds for all  $k$ , then there exists  $\gamma \in \mathcal{J}_{m, 1}^{r, \infty}$  with  $\lambda_x^{(\infty)}(\gamma) = j_x^\infty g$ . Just let  $\gamma$  be such that  $\pi_k^\infty(\gamma) = \gamma_k$  for each  $k$ . But note that for  $\gamma \in \mathcal{J}_{m, 1, x}^{r, \infty}$ , the Borel Lemma, Lemma 5.1, implies that there exists  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  $j_x^{r, \infty} f = \gamma$ .  $\square$

**Notation.** If  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , we will denote

$$(28) \quad {}^*j_x^\sigma(f) = ({}^*(j_{*x}^k(f)))_{k \in \mathbb{N}_0}, \text{ an external sequence.}$$

Similarly, if  $\lambda$  is an internal jet map and we are considering, for each  $k \in \mathbb{N}$ , not  ${}^*\mathbb{N}$ ,  $\lambda_{*x}^{(k)}$ , the **internal** prolongation of  $\lambda$  at the standard point  ${}^*x$ , ie.,  $(\lambda_{*x}^{(k)})_{k \in {}^\sigma\mathbb{N}}$ , then we will also write this as  $\lambda_x^{(\sigma)}$ ; eg., if  $\lambda$  or  $f$  are standard and we are considering only this family of internal prolongations of  ${}^*\lambda$  or  ${}^*f$ , then we will write  ${}^*j_x^\sigma({}^*f)$  or  ${}^*\lambda_x^{(\sigma)}$ .

In the situation when  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$ ,  ${}^*j_x^\sigma({}^*f)$  is just the external sequence of standard numbers,  $({}^*(j_x^k f))_{k \in \mathbb{N}_0}$ . This notation can be unwieldy; some of the parentheses, or  ${}^*$ 's may be left out if the meaning is still clear. Note that if

$$(29) \quad \mathcal{V}^m = \bigcap_{x \in \mathbb{R}^m} \mathcal{V}_x^m = \{f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R}) : {}^*j_x^\sigma(f) = 0, \text{ for all } x \in \mathbb{R}^m\},$$

then  $\mathcal{V}^m < {}^\sigma\mathcal{J}_\lambda$ . Although  $\mathcal{V}$  is a  ${}^*\mathbb{R}$  vector space, it is nonetheless external. To get a sense of the size of  $\mathcal{V}^m$  in  ${}^*C^\infty(M, \mathbb{R}^n)$ , note that  $\mathcal{L} < \mathcal{V}$  where  $\mathcal{L}$  is the  ${}^*$ finite codimensional subspace  ${}^*C^\infty(M, \mathbb{R}^n)$  defined in the concluding section of the paper. Therefore, we have the following consequence of Lemma 5.4.

**Corollary 5.2.** Suppose that  ${}^\sigma\mathcal{J}_{*\lambda, *x} \neq 0$  for some  $x \in \mathbb{R}^m$ . Then  ${}^\sigma\mathcal{J}_\lambda$  is  ${}^*$ infinite dimensional.

**Remark.** Suppose that  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$ ,  $\bar{f} \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that for some standard  $x$ , and  $0 < \delta \sim 0$ ,  $\bar{f}|_{D_\delta(*x)} = {}^*f|_{D_\delta(*x)}$ . Then the internal jet sequence  ${}^*j_{*x}^\infty \bar{f} \doteq ({}^*j_{*x}^k \bar{f})_{k \in {}^*\mathbb{N}}$  is just the  ${}^*$ transfer of the standard sequence  $(j_x^k f)_{k \in \mathbb{N}_0}$ , eg., when the set of jet indices is restricted to to the external set  ${}^\sigma\mathbb{N}_0$ . That is, in the above notation,  ${}^*j_x^\sigma(\bar{f}) = ({}^*j_x^k f)_{k \in \mathbb{N}_0}$ .

**5.2. Many generalized solutions with high contact.** The following result is the main linear result of the paper, although its import is not apparent without the following corollaries.

**Theorem 5.1.** *Suppose that  $P \in LPDO_r$ . Then, for every  $g \in {}^\sigma\mathfrak{I}_\lambda$ , there exists  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that*

$$(30) \quad {}^*j_x^\sigma({}^*P(f)) = {}^*j_x^\sigma g \text{ for every } x \in \mathbb{R}^m.$$

*That is,  ${}^*P(f)$  has  ${}^\sigma$ infinite order  ${}^*$ contact with  $g$  at all points of  ${}^*\mathbb{R}_{nes}^m$ .*

*Proof.* Suppose that  ${}^\sigma\mathfrak{I}_\lambda$ . By Lemma 5.5 if  $x \in \mathbb{R}^m$ , there is  $f^x \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that

$$(31) \quad \lambda_{P,x}^{(\infty)}(j_x^\infty f^x) = j_x^\infty g.$$

By Lemma 5.2 there exist  $\bar{f} \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that for every  $x \in \mathbb{R}^m$   $\bar{f}|_{D_\delta(*x)} = {}^*f^x|_{D_\delta(*x)}$ . By the remark above, for each such standard  $x$ ,  ${}^*j_x^\sigma \bar{f} = {}^*j_{*x}^\sigma({}^*f^x)$ . But this implies that, at each standard  $x$ ,

$$(32) \quad {}^*\lambda_{*x}^{(\sigma)}({}^*j_{*x}^\sigma \bar{f}) = {}^*\lambda_{*x}^{(\sigma)}({}^*j_{*x}^\sigma {}^*f^x)$$

Coupling this with the transfer of expression (31) restricted to standard indices, we now have  $\bar{f} \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that

$$(33) \quad {}^*\lambda_{*x}^{(\sigma)}({}^*j_{*x}^\sigma \bar{f}) = {}^*j_{*x}^\sigma g$$

for each  $x \in \mathbb{R}^m$ . But, by definition of prolongation,  ${}^*$ transferred

$$(34) \quad {}^*j_{*x}^\infty({}^*P(\bar{f})) = {}^*\lambda_{*x}^{(\infty)}({}^*j_{*x}^{r,\infty} \bar{f}).$$

Stringing together expressions (33) and (34), restricted to standard indices, gets our result, as this holds for every standard  $x$ .  $\square$

**Corollary 5.3.** *Suppose that  $P \in LPDO_r$  with symbol  $\lambda$ , and principal symbol  $\underline{\lambda}$ . Suppose that for each  $x \in \mathbb{R}^m$ ,  $\underline{\lambda}_x \neq 0$ . Then for every  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , there exists  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  with*

$$(35) \quad {}^*j_{*x}^\infty({}^*P(f)) = {}^*j_{*x}^\infty g \text{ for every } x \in \mathbb{R}^m.$$

*Proof.* By Lemma 4.1, if  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$ , then  $g \in {}^\sigma\mathfrak{I}_\lambda$ . But then the result is a direct consequence the previous theorem.  $\square$

**Remark.** To put this result in perspective, note that Todorov, [27], proves the  $0^{th}$  order jet case in his paper, with a slightly weaker hypothesis.

For those  $x \in \mathbb{R}^m$  where  $\lambda_x = 0$ , a trivial case for the 0-jet, as Todorov notes, becomes a nontrivial thickened result when the consideration becomes the infinite jet at standard points where some finite prolongation  $\lambda_x^{(k)}$  is nonzero. For this situation we have the following result.

**Corollary 5.4.** *Suppose that  $\text{finsupp}(P) = \mathbb{R}^m$ . Then there exists an  ${}^*$ infinite dimensional subspace  ${}^\sigma\mathfrak{I}_P < {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that if  $g \in {}^\sigma\mathfrak{I}_P$ , then there exists  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that*

$$(36) \quad {}^*j_{*x}^\sigma({}^*P(f)) = {}^*j_{*x}^\sigma g \text{ for every } x \in \mathbb{R}^m.$$

*Proof.* As  $f \text{insupp}(P) = \mathbb{R}^m$ , if  $x \in \mathbb{R}^m$ , Lemma 5.3 implies that  $\mathcal{S}_x^P \doteq \{j_x^\infty g : g \in \mathfrak{I}_{\lambda,x}\}$  is nonzero. Therefore, the result follows from Corollary 5.2 and the above theorem.  $\square$

That is, even if the symbol vanishes at points of  $\mathbb{R}^m$ , as long as this vanishing order is finite at each such point, then there exists many  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , satisfying the above compatibility conditions, such that  ${}^*P(f) = g$  is solved to infinite order along  ${}^\sigma\mathbb{R}^m$  by  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ .

**5.3. Solutions for singular Lewy operator.** Before we move to the next section, let's look at the Lewy operator, see [27], p.679,  $\mathcal{L} = \partial_1 + i\partial_2 - 2i(x_1 + ix_2)\partial_3$  acting on smooth complex valued functions on  $\mathbb{R}^3$ . First of all, note that the results just proved hold just as well with complex valued functions; the proofs are identical. Second, note that the principal symbol,  $\lambda_{\mathcal{L}}$  of  $\mathcal{L}$  is the same as the total symbol  $\lambda_{\mathcal{L}} = y_1 + iy_2 - 2i(x_1 + ix_2)y_3$ . Inspection shows that these maps are nonvanishing, hence  $\mathcal{L}$  satisfies the hypotheses in Corollary 5.3, ie., for any  $g \in {}^*C^\infty(\mathbb{R}^3, \mathbb{C})$ , there exists (many)  $f \in {}^*C^\infty(\mathbb{R}^3, \mathbb{C})$  such that  ${}^*\mathcal{L}(f)(^*x) = {}^*g(^*x)$  to infinite order at each  $x \in \mathbb{R}^3$ . But we can say more. Suppose that  $h = (h_1, h_2, h_3)$  is such that  $h_i \in C^\infty(\mathbb{R}^3, \mathbb{R})$  for each  $i$  and  $h$  vanishes to finite order at each  $x \in \mathbb{R}^3$ . Let  $\hat{\mathcal{L}} = h_1(x)\partial_1 + ih_2(x)\partial_2 - 2ih_3(x)(x_1 + ix_2)\partial_3$ , a kind of singular Lewy operator with finite singularities at each  $x \in \mathbb{R}^3$ . Then Corollary 5.4 implies that for any  $g \in C^\infty(\mathbb{R}^3, \mathbb{C})$  that vanishes where  $\lambda_{\hat{\mathcal{L}}}$  vanishes to order at least that of  $h$ , there exists  $f \in {}^*C^\infty(\mathbb{R}^3, \mathbb{C})$  such that  $\hat{\mathcal{L}}(f)(^*x) = g(^*x)$  holds to infinite order at all  $x \in \mathbb{R}^3$ .

## 6. NONLINEAR PDE'S AND THE POINTWISE LIFTING PROPERTY

In this section  $P$  can now be an arbitrary smooth nonlinear PDO of finite order. Only the rudiments of a nonlinear development parallel to the linear considerations in the previous sections will be attempted in this paper. The point here is that the framework is not an impediment to a consistent consideration of generalized objects.

First, as it is natural within our framework, we straightforwardly extend the notion of solution of a differential equation, as defined in Todorov's paper, to include nonlinear as well as linear differential equations. In analogy with  $LPDO_r$ , a (possibly nonlinear) order  $r$  partial differential operator,  $P : C^\infty(\mathbb{R}^m, \mathbb{R}) \rightarrow C^\infty(\mathbb{R}^m, \mathbb{R})$ , is a mapping given by  $P(f)(x) = \lambda(j_x^r f)$  where now the total symbol of  $P$ ,  $\lambda : \mathcal{J}_{m,1}^r \rightarrow \mathbb{R}_m$  is a possibly nonlinear smooth bundle map. Let  $\mathbf{NLDO}_r$  denote this set of operators.

**Definition 6.1.** Given  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , we say that  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  is a solution of  ${}^*P(f) = g$  if  ${}^*P(f)(^*x) = g(^*x)$  for every  $x \in \mathbb{R}^m$ .

We will consider a simple set theoretic condition on pairs  $(P, g)$  (or  $(\lambda_P, g)$ ) the **pointwise covering property**, **PCP**. An easy (saturation) proof will get that if  $(P, g)$  satisfies this property, written  $(P, g) \in PCP$ , or  $(\lambda_P, g) \in PCP$ , then  $P(f) = g$  has generalized solutions in a sense of Todorov. We will then show that the main theorem is a corollary of this result by verifying that our linear differential equation satisfies **PCP**.

**Definition 6.2.** Let  $\lambda \in C^\infty(\mathcal{J}_{m,1}^k, \mathbb{R})$  and  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$ . We say that the pair  $(\lambda, g)$  satisfies **PCP**, if for each  $x \in \mathbb{R}^m$ , there exists  $p \in (\pi^k)^{-1}(x)$ , such that

$\lambda(p) = g(x)$ . If  $\lambda \in {}^*C^\infty(\mathcal{J}_{m,1}^k, \mathbb{R})$  and  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , then we say that **the pair  $({}^*\lambda, g)$  satisfies  ${}^\sigma PCP$**  if for all  $x \in {}^\sigma\mathbb{R}^m$ , there exists  $p \in {}^*\pi^{-1}(x)$ , such that  $\lambda(p) = g({}^*x)$ .

**Remark.** Note that finding  $p \in (\pi^r)^{-1}(x)$  such that  $\lambda(p) = g(x)$  is identical to finding  $h \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  $h$  solves  $P(h) = g$  at the single point  $x$ . Also, note the relationship between  $PCP$  and  ${}^\sigma PCP$ . If  $\lambda \in C^\infty(\mathcal{J}_{m,1}^k, \mathbb{R})$  and  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$  are such that  $(\lambda, g) \in PCP$ , then  $({}^*\lambda, {}^*g) \in {}^\sigma PCP$ . On the other hand, if  $\lambda \in SC^\infty(\mathcal{J}_{m,1}^k, \mathbb{R})$  and  $g \in SC^\infty(\mathbb{R}^m, \mathbb{R})$  are such that  $(\lambda, g) \in {}^\sigma PCP$ , then  $({}^\circ\lambda, {}^\circ g) \in PCP$ . (Recall that if  $X, Y$  are Hausdorff topological spaces and  $f : {}^*X \rightarrow {}^*Y$  is such that  $f$  maps nearstandard points of  ${}^*X$  to those of  ${}^*Y$ , then the standard part of  $f$ ,  ${}^\circ f : X \rightarrow Y$  is a welldefined map.)

The following lemma verifies that the the  $PCP$  condition restricted to linear differential operators has Todorov's criterion as a special case. We are working with the symbol of the operator.

**Lemma 6.1.** *Let  $P \in LPDO_r$  and write*

$$\lambda_P = \sum_{|\alpha| \leq r} f_\alpha y_\alpha.$$

*Suppose that  $\sum_{|\alpha| \leq r} |f^\alpha(x)| \neq 0$  for all  $x \in \mathbb{R}^m$ . Then for all  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ ,  $({}^*\lambda_P, g) \in {}^\sigma PCP$ .*

*Proof.* Let  $x_0 \in \mathbb{R}^m$ . We will not write  ${}^*x_0$  when we transfer. The condition guarantees that there exists a multiindex  $\alpha$  such that  $f^\alpha(x_0) \neq 0$ . Let  $\Gamma = \{\alpha : c_\alpha \doteq f^\alpha(x_0) \neq 0\}$ . If  $\Gamma$  has only one element,  $\alpha_0$ , let  $h \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  be such that

$$(37) \quad {}^*\partial^{\alpha_0}(h)(x_0) = \frac{g(x_0)}{{}^*f^{\alpha_0}(x_0)}$$

Then if  $\kappa \in {}^*\mathcal{J}_{m,1}^k$  is given by  ${}^*j_{x_0}^r h$ , we get that

$$(38) \quad \begin{aligned} {}^*\lambda_P(\kappa) &= {}^*\lambda_P({}^*j_{x_0}^r(h)) = \\ &= {}^*\sum_{|\alpha| \leq r} f^\alpha(x_0) y_\alpha(j_{x_0}^r h) = f^{\alpha_0}(x_0) y_{\alpha_0}(j_{x_0}^r h) = \\ &= f^{\alpha_0}(x_0) \frac{g(x_0)}{f^{\alpha_0}(x_0)} = g(x_0) \end{aligned}$$

as we wanted. So suppose that  $\Gamma$  has at least two elements. Let  $\alpha_0 \in \Gamma$  and let  $\Lambda = \Gamma - \{\alpha_0\}$ . Choose  $h \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  so that if  $\alpha \in \Lambda$ , then  ${}^*\partial^\alpha h(x_0) = 0$  and (as in the first case) such that  ${}^*\partial^{\alpha_0}(h)(x_0) = \frac{g(x_0)}{f^{\alpha_0}(x_0)}$ . Then as in expressions (38), we get  $P(h)(x_0) = g(x_0)$ .  $\square$

Given the above lemma we shall see that Todorov's result is a corollary of this lemma and the next theorem proving the existence of solutions of  $PCP$  operators. Before we proceed to the theorem, we need some NSA preliminaries. First we give a simple example of the construction we will need. Let  $F(\mathbb{R})$  be all maps from  $\mathbb{R}$  to  $\mathbb{R}$  and let  $F^\infty(\mathbb{R}) = \{f \in F(\mathbb{R}) : f \text{ is smooth}\}$ . Let  $f \in F(\mathbb{R})$ , then there exists an (internal) element  $\tilde{f} \in {}^*F^\infty(\mathbb{R})$  such that  $\tilde{f}|_{{}^\sigma\mathbb{R}} = f$ , as the following argument shows. Let  $\mathcal{Y}_1 \subset {}^*\mathbb{R}$  be  ${}^*$ finite such that  ${}^\sigma\mathbb{R} \subset \mathcal{Y}_1$  and let  $\mathcal{Y}_2 = {}^*f(\mathcal{Y}_1)$ .

Then  $\mathcal{Y}_2$  is obviously a  $^*\text{finite}$  subset of  $^*\mathbb{R}$ . Now consider the following elementary standard statement. If  $S_1, S_2$  are finite subsets of  $\mathbb{R}$ , of the same cardinality, and  $h : S_1 \rightarrow S_2$  is a bijection, there exists  $\tilde{h} \in F^\infty(\mathbb{R})$  such that  $\tilde{h}|_{S_1} = h$ . This follows from a simple partition of unity argument. Now  $^*\text{transfer}$  this to get existence of  $\tilde{f} \in {}^*F^\infty(\mathbb{R})$  such that  $\tilde{f}|_{\mathcal{Y}_1} = {}^*f|_{\mathcal{Y}_1}$ . In particular,  $\tilde{f}|_{\sigma\mathbb{R}} = {}^*f|_{\sigma\mathbb{R}} = f|_{\mathbb{R}}$ , as we wanted. Now we want to do the same construction in the venue of bundles and their sections. Let  $\Gamma(\mathcal{J}_{m,1}^r) = \{s : \mathbb{R}^m \rightarrow \mathcal{J}_{m,1}^r \mid \pi^r \circ s = \mathbb{I}_{\mathbb{R}^m}\}$ , ie., set theoretic sections of  $\pi^r$ . Let  $\Gamma^\infty(\mathcal{J}_{m,1}^r) = \{s \in \Gamma(\mathcal{J}_{m,1}^r) : s \text{ is a smooth map}\}$ . We have the following lemma.

**Lemma 6.2.** *Suppose that  $s \in \Gamma(\mathcal{J}_{m,1}^r)$ . Then there exists  $\tilde{s} \in {}^*\Gamma^\infty(\mathcal{J}_{m,1}^r)$ , such that  $\tilde{s}|_{\sigma\mathbb{R}^m} = s|_{\mathbb{R}^m}$ .*

*Proof.* As with the above example, let  $\mathcal{X} \subset {}^*\mathbb{R}^m$  be  $^*\text{finite}$  such that  $\mathbb{R}^m \subset \mathcal{X}$ . We have the following elementary fact. If  $B = \{b_1, \dots, b_l\}$  is a finite subset of the base and  $P = \{p_1, \dots, p_l\} \subset \mathcal{J}_{m,1}^r$  is a finite subset such that  $p_j \in (\pi^r)^{-1}(b_j)$  for each  $j$ , then there exists  $s \in \Gamma(\mathcal{J}_{m,1}^r)$  such that  $s(x_j) = p_j$  for all  $j$ . Now  $^*\text{transfer}$  this statement, applying the  $^*\text{transferred}$  statement to the  $^*\text{finite}$  subset  $\mathcal{X}$  in the base and the  $^*\text{finite}$  subset  ${}^*s(\mathcal{X})$  of points in the  $^*\text{bundle}$  over  $\mathcal{X}$ . That is, we can infer the existence of an internal section  $\tilde{s} \in {}^*\Gamma^\infty(\mathcal{J}_{m,1}^r)$  such that for all  $x \in \mathcal{X}$ ,  $\tilde{s}(x) = {}^*s(x)$ , in particular  $\tilde{s}|_{\sigma\mathbb{R}^m} = s$ , as we wanted.  $\square$

In the context of this lemma, we have that  $({}^*\lambda, g) \in {}^\sigma PCP$  is equivalent to the existence of a set theoretic section  $s \in {}^*\Gamma(\mathcal{J}_{m,1}^r)$  such that the pointwise condition  ${}^*\lambda \circ {}^*s = g$  holds on  $\sigma\mathbb{R}^m$ . It's important to note that, generally speaking, such sections are far from integrable; that is, equal to  $j^r f$  for some smooth  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$ . But again, by a transfer argument, we can find such a section.

**Lemma 6.3.** *Suppose that  $s \in \Gamma^\infty(\mathcal{J}_{m,1}^r)$  and  $\mathcal{X} \subset {}^*\mathbb{R}^m$ . Then there exists  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  ${}^*j^r f|_{\mathcal{X}} = s|_{\mathcal{X}}$ .*

*Proof.* This just follows from the  $^*\text{transfer}$  of the following obvious standard statement about jets. If  $\{p_1, p_2, \dots, p_l\} \subset \mathcal{J}_{m,1}^r$  such that  $x_j = \pi^r(p_j)$  are all distinct. Then there exists  $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$  such that  $j_{x_i}^r f = p_i$  for all  $i$ .  $\square$

With these preliminaries, the proof of the following result is immediate.

**Theorem 6.1.** *Let  $\mathcal{D} \in NLDO_r$  and let  $\lambda_{\mathcal{D}} = \lambda \in C^\infty(\mathcal{J}_{m,1}^r, \mathbb{R})$  and  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ . Suppose that  $({}^*\lambda, g) \in {}^\sigma PCP$ . Then  $\mathcal{D}(f) = g$  has a generalized solution,  $f$ , in the sense of Todorov.*

*Proof.* By the remark above,  $({}^*\lambda, g) \in {}^\sigma PCP$  is equivalent to the existence of an  $s \in {}^*\Gamma(\mathcal{J}_{m,1}^r, \mathbb{R})$  such that for every  $x \in \mathbb{R}^m$ ,  $\lambda_{*_{\mathcal{D}}}(s({}^*x)) = g({}^*x)$ . But by Lemma 6.2, there exists  $\tilde{s} \in {}^*\Gamma^\infty(\mathcal{J}_{m,1}^r)$  such that  $\tilde{s}({}^*x) = s(x)$  for all  $x \in \mathbb{R}^m$ . And by Lemma 6.3, there exists  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , such that for all  $x \in \mathbb{R}^m$ ,  ${}^*j_{*x}^r(f) = \tilde{s}({}^*x)$ .  $\square$

Todorov's existence result (being for linear operators only) is a special consequence of the previous development.

**Corollary 6.1.** *Suppose that  $g \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$  and  $P \in LPDO_r$  is such that  $\lambda_P$  is nonvanishing on  $\mathbb{R}^m$ . Then there exists  $f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R})$ , such that for all  $x \in \mathbb{R}^m$ ,  $P(f)({}^*x) = g({}^*x)$ , ie.,  $f$  is a solution of  $P(f) = g$  in the manner of Todorov.*

*Proof.* This is clear.  $\square$

Given the nonlinear setting of this section, proving results analogous to those in the linear sections appear to need much more involved preliminaries and so will be pursued at a later date. Nonetheless, it seems clear that we can consider some general criteria revolving around when  $(P, g) \in PCP$ . In particular, it appears that we can prove a *universal existence theorem asserting that any possible space of generalized functions that has the PCP property is already contained in our nonstandard space*. This, too, will appear as time allows.

## 7. CONCLUSION

**7.1. Too many solutions?** In this paper I have used some of the machinery of the geometry of partial differential equations to explore the possibilities of the approach of Todorov. (We have yet to work through the nonlinear analogs of the linear results presented here; this will entail a much more extensive use of the the jet theory of nonlinear partial differential operators. Note even more starkly than in this paper; no counterpart in standard mathematics exists.) The implications of the results of this paper are still not clear. Yet one thing should be obvious, the class of internally smooth maps are remarkably ‘flabby’, as compared to the standard world.

As an indication of this, we have the following construction. let  $\mathcal{L} \subset {}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  be the  ${}^*\mathbb{R}$  linear subspace of  ${}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  defined as follows. Let  $\mathcal{Y} \subset {}^*\mathbb{R}^m$  be a  ${}^*$ finite subset such that  ${}^\sigma\mathbb{R}^m \subset \mathcal{Y}$ . Let  $\omega \in {}^*\mathbb{N}_\infty$ . Then, the set  $\mathcal{L} = \{f \in {}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n) : {}^*j_x^\omega f = 0 \text{ for all } x \in \mathcal{Y}\}$  is a  ${}^*$ cofinite dimensional subspace of  ${}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ , as this set of conditions on elements  $f$  in  ${}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  given by specifying the value of  $j_x^\omega f$ , a  ${}^*$ finite number of  ${}^*$ Taylor coefficients at a  ${}^*$ finite set of points in  ${}^*\mathbb{R}^m$ , ie., the points of  $\mathcal{Y}$  is  ${}^*$ finite. Now, by construction,  $\mathcal{L} \cap {}^\sigma C^\infty(\mathbb{R}^m, \mathbb{R}^n) = \{0\}$ , and we have the following diagram

$$(39) \quad \begin{array}{ccccc} & \mathcal{L} & & & \\ & \downarrow j & & & \\ {}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n) & \xrightarrow{{}^*j^\omega} & {}^*\mathcal{J}_{m,n}^\omega & \xrightarrow{\rho} & {}^*\mathcal{J}_{m,n}^\omega|_{{}^\sigma\mathbb{R}^m} \\ & \uparrow i & & & \\ {}^*\mathbb{R} \otimes {}^\sigma C^\infty(\mathbb{R}^m, \mathbb{R}^n) & & & & \end{array}$$

where the maps  $i$  and  $j$  are  ${}^*\mathbb{R}$  subspace injections and  $\rho$  is the highly external restriction to the fibers over  ${}^\sigma\mathbb{R}^m$ . Let  $\Phi = \rho \circ {}^*j^\omega$ . Then the following holds.

**Lemma 7.1.**  $\Phi|_{Im(j)}$  has image  $\{0\}$  and  $\Phi|_{Im(i)}$  is an injection.

*Proof.* By construction, we have  $\Phi(f) = 0$  for every  $f \in \mathcal{L}$ . On the other hand, if for any element  $f \in {}^\sigma C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  we have  ${}^*j_x^\omega({}^*f) = 0$  for each  $x \in \mathcal{Y}$ , we have in particular that  $j_x^\infty f = 0$ , for each  $x \in \mathbb{R}^m$ , that is  $f = 0$ . This therefore holds for all  $f \in {}^*\mathbb{R}^m \otimes_{{}^\sigma\mathbb{R}^m} {}^\sigma C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ .  $\square$

So we have that the subspace of elements of  ${}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  whose  ${}^\sigma$  infinite  ${}^*$ jet vanishes everywhere on  ${}^\sigma\mathbb{R}^m$  is all of  ${}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  up to a  ${}^*$ finite dimensional subspace containing all standard smooth maps. It should therefore be clear that we have the immediate corollary that exemplifies the ability to bend almost all of



${}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  away from contact with the world of standard differential equations, at least at standard points.

**Corollary 7.1.** *If  $P \in NPDO_r$  for any  $r \in \mathbb{N}$  such that  $P(\text{zero map}) = \text{zero map}$ , then  ${}^*P(f)({}^*x) = 0$  for all  $f \in \mathcal{L}$  and all  $x \in \mathbb{R}^m$ .*

*Proof.* All classical differential operators  $P$  of order  $r$ , factor as  ${}^*P = {}^*\lambda_P \circ {}^*j^r = {}^*\lambda_P \circ {}^*\pi_r^\omega \circ {}^*j^\omega$  and by above  $j^\omega(\mathcal{L})|{}^\sigma\mathbb{R}^m = \{0\}$ .  $\square$

*That is, all classical partial differential operators sending the zero map to the zero map operate as zero maps on “almost all” of  ${}^*C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ .*

One perspective on the results here should not be a surprise: that  ${}^*$ smooth functions (and with some thought  ${}^*$ analytic functions) are far too flabby on a full infinitesimal scale. From a positive viewpoint, one could see how this might allow an investigator to have wide latitude in ‘Tayloring’ generalized functions (on the monadic level) to get appropriate rigidities-growth or to test various empirical results by infinitesimal adjustments of singular parameters. The remark (in the introduction) with respect to the work of Baty, et al, see eg., [4] seems relevant to the second perspective. The algebras of Oberguggenberger and Todorov, [20] and the further developments in eg., Todorov and Vernaev, [25] seems to be good examples of the Tayloring capacities.

**7.2. Prospects and goals.** Only the rudiments of jets on the one hand, and non-standard analysis, on the other have been deployed in this paper. In follow up articles we intend to use ( ${}^*$ transferred) tools from smooth function theory along with a more extensive use of jet theory to extend both the linear and nonlinear existence results. Further, deploying more nuanced version of the jet material of section 4 over certain types of infinite points in the jet fibers, we intend to prove results on regularity of solutions of partial differential operators, linear or nonlinear, whose symbols satisfy certain properness conditions. Our first paper along this line, [17], gives a regularity theorem for a broad class of nonlinear differential operators. We also intend to extend the results here to include the results of Akiyama, [1] into the framework established here in the manner we have included the results of Todorov. The method is by an extension from internal mapping with  ${}^*$ finite support to internal smooth modules of bundle sections with  ${}^*$ finite support. Furthermore, as noted in the introduction, we will refine the arguments in this paper to Todorov’s nonstandard Colombeau algebras. Given that all of the usual constructions on the symmetries of differential equations (as in eg., Olver, [21]) are straightforwardly lifted to the nonstandard universe, we are also looking into developing a theoretic framework on generalized symmetries (eg., shock symmetries) of differential equations, continuing within the jet theoretic framework begun here.

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